

MIND MAP

MATHEMATICS





CONTENTS

MATHEMATICS 63-98

❖ Sets.....	65
❖ Trigonometric Ratios and Equations	66-67
❖ Quadratic Equation	68-69
❖ Complex Number	70-71
❖ Linear Inequalities & Linear Programming Problems	72
❖ Permutation & Combination.....	73-74
❖ Binomial Theorem.....	75-76
❖ Sequence & Series	77-78
❖ Straight Line	79-80
❖ Circle	81-84
❖ Parabola	85-87
❖ Ellipse	88-90
❖ Hyperbola	91-94
❖ Statistics.....	95
❖ Mathematical Reasoning & PMI	96
❖ Properties and Solution of a Triangle	97-98

SETS

01

Introduction

A set is collection of well-defined distinguished objects.

By well defined we mean that there should be no ambiguity regarding the inclusion and exclusion of the objects.

For example a collection of scariest movies can't be considered as a set, because it will differ from person to person

02

Cardinal Number

The number of elements in a finite set is represented by $n(A)$, known as cardinal number.

Eg. $A = \{a, b, c, d, e\}$ Then, $n(A) = 5$

03

Subset

Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B and written as $A \subseteq B$ or $B \supseteq A$ (read as 'A' is contained in 'B' or 'B' contains 'A'). B is called superset of A.

Note:

- Every set is a subset and superset of itself.
 - If A is not a subset of B, we write $A \not\subseteq B$.
 - The empty set is the subset of every set.
 - If A is a set with $n(A) = m$, then no. of subset of A are 2^m and the number of proper subsets of A are $2^m - 1$
- Eg. Let $A = \{3, 4\}$, then subsets of A are $\phi, \{3\}, \{4\}, \{3, 4\}$. Here, $n(A) = 2$ and number of subsets of $A = 2^2 = 4$

04 Types of Sets

Empty set or Null set

A set which has no element is called null set. It is denoted by symbol ϕ or $\{\}$.

Equivalent set

Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$. Clearly, equal set are equivalent but equivalent set need not to be equal.

Singleton set

A set having one element is called singleton set.

Finite and Infinite set

A set which has finite number of elements is called a finite set. Otherwise, it is called an infinite set.

Power set

The set of all subset of a given set A is called power set of A and denoted by $P(A)$.

Equal set

Two sets A and B are said to be equal, written as $A = B$, if every element of A is in B and every element of B is in A.

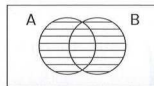
05 Operations on Sets

Difference of two sets

If A and B are two sets, then their difference A-B is defined as:
 $A-B = \{x : x \in A \text{ and } x \notin B\}$ Similarly,
 $B-A = \{x : x \in B \text{ and } x \notin A\}$

Union

The union of two sets A and B, written as $A \cup B$ (read as A union B) is the set of all elements which are either in A or in B or in both. Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

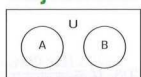


clearly, $x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$ and
 $x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B$

Symmetric Difference

The Symmetric difference of two sets A and B, denoted by $A \Delta B$, is defined as $(A \Delta B) = (A - B) \cup (B - A)$

Disjoint sets



Two sets A and B are said to be disjoint, if $A \cap B = \phi$ i. e. A and B have no common element.

Intersection

The intersection of two sets A and B, written as $A \cap B$ (read as 'A' intersection 'B') is the set consisting of all the common elements of A and B.



Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$
Clearly, $x \in A \cap B \Rightarrow \{x \in A \text{ and } x \in B\}$ and
 $x \notin A \cap B \Rightarrow \{x \notin A \text{ or } x \notin B\}$.

Compliment of set

If U is a universal set and A is a subset of U, then complement of A is the set which contains those elements of U, which are not present in A and is denoted by A' or A^c . Thus, $A^c = \{x : x \in U \text{ and } x \notin A\}$

Properties of complement

Complement law:

(i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

De Morgan's Law:

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Double Complement law:
 $(A')' = A$

Law of empty set and universal set
 $\phi' = U$ and $U' = \phi$

Results on Operation of Sets:

1. $A \subseteq A \cup B, B \subseteq A \cup B, A \cap B \subseteq A, A \cap B \subseteq B$
2. $A - B = A \cap B^c$
3. $(A \cup B) \cap (A \cup B)' = A$
4. $(A - B) \cup B = A \cup B$
5. $(A - B) \cap B = \phi$
6. $A \subseteq B \Leftrightarrow B' \subseteq A'$
7. $A - B = B' - A'$
8. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$
9. $A - B = B - A \Leftrightarrow A = B$
10. $A \cup B = A \cap B \Leftrightarrow A = B$

Cardinal Number of Some Sets:

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. $n(A \cap B) = n(A) - n(A - B)$
4. $n(A \cup B) = n(A) + n(B)$. [If A and B disjoint]
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A - B) - n(A' \cap B)$
9. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
10. $n(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = n(A_1) + n(A_2) + \dots + n(A_n)$.
[If $A_1, A_2, \dots, A_n$ are disjoint sets]

07

Cartesian Product

$A \times B = \{(a, b) : a \in A, b \in B\}$.

$A \times B \neq B \times A$.

If A has p elements & B has q elements then $A \times B$ has pq elements.

08

Important results on Cartesian product:

If A, B, C are three sets.

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$
5. If $A \subseteq B, (A \times C) \subseteq (B \times C)$.
6. If $A \subseteq B, (A \times B) \cap (B \times A) = A \times A$.
7. If $A \subseteq B$ & $C \subseteq D$, then $A \times C \subseteq B \times D$.

09

Congruence

Let m be a positive integer, then the two integers a & b are said to be congruent modulo m if a - b is divisible by m. i.e. $a - b = m \cdot k$.
 $a \equiv b \pmod{m}$.

TRIGONOMETRIC RATIOS AND EQUATIONS

01 IMPORTANT TRIGONOMETRIC RATIOS

- $\sin 15^\circ$ or $\sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos \frac{5\pi}{12}$
- $\cos 15^\circ = \cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \sin \frac{5\pi}{12}$
- $\sin \frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$
- $\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75^\circ$
- $\cos \frac{\pi}{5}$ or $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$
- $\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15^\circ$
- $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$
- $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$, $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$
- $\tan 15^\circ = 2 - \sqrt{3}$, $\tan 22.5^\circ = \sqrt{2} - 1$, $\tan 67.5^\circ = \sqrt{2} + 1$,
- $\tan 75^\circ = 2 + \sqrt{3}$.

04 Transformation of Product into Sum or Difference of Sines or Cosines

01 $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

02 $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

03 $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

04 $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

05 Some Special Series

01

$$\sin \alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin\{\alpha + (n-1)\beta\} = \frac{\sin\left\{\frac{2\alpha + (n-1)\beta}{2}\right\} \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}$$

02

$$\cos \alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos\{\alpha + (n-1)\beta\} = \frac{\cos\left\{\frac{2\alpha + (n-1)\beta}{2}\right\} \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}$$

02 Trigonometrical Equations

- $\cos n\pi = (-1)^n$, $\sin n\pi = 0$, $n \in I$
- $\cos \frac{n\pi}{2} = 0$, $\sin \frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, n is odd integer.
- $\cos(n\pi + \theta) = (-1)^n \cos \theta$, $n \in I$
 $\sin(n\pi + \theta) = (-1)^n \sin \theta$, $n \in I$
- $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin \theta$, n is odd integer
- $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos \theta$, n is odd integer

03 Factorisation of the sum or Difference of sine and cosine with two variables

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
- $\sin 2\theta = 2\sin \theta \cos \theta = \frac{2\tan \theta}{1+\tan^2 \theta}$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}$
- $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$, $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$, $\tan^2 \theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$
- $\tan 2\theta = \frac{2\tan \theta}{1-\tan^2 \theta}$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$, $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$, $\tan 3\theta = \frac{3\tan \theta - \tan^3 \theta}{1-3\tan^2 \theta}$
- $\sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$, $\sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2}$
- $\cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2}$, $\cos C - \cos D = 2\sin \frac{C+D}{2} \sin \frac{D-C}{2}$
- $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$, $\tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}$

06 QUICK LOOK

For any three angles A, B, C

- $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B + \sin A \sin B \sin C$
- $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
- $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
- $\tan(A+B+C+D+\dots) = \frac{S_1 - S_3 + S_5 - \dots}{1 - S_2 + S_4 - \dots}$
- $S_1 = \sum \tan A$, $S_2 = \sum \tan A \tan B, \dots$

- $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
- $\tan(nA) = \frac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \dots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A - \dots}$
- $\sin(B+C) = \sin A, \cos B = -\cos(C+A)$ for ΔABC
- $\cos(A+B) = -\cos C, \sin C = \sin(A+B)$ for ΔABC
- $\tan(C+A) = -\tan B, \cot A = -\cot(B+C)$ for ΔABC
- $\cos \frac{A+B}{2} = \sin \frac{C}{2}, \cos \frac{C}{2} = \sin \frac{A+B}{2}$ for ΔABC
- $\sin \frac{C+A}{2} = \cos \frac{B}{2}, \sin \frac{A}{2} = \cos \frac{B+C}{2}$ for ΔABC
- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ for ΔABC
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$ for ΔABC
- $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ for ΔABC
- $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ for ΔABC
- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ for ΔABC
- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ for ΔABC
- $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$ for ΔABC
- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$ for ΔABC
- $\sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1} \cdot 4 \sin mA \sin mB \sin mC$
- $\cos mA + \cos mB + \cos mC = 1 \pm 4 \sin \frac{mA}{2} \sin \frac{mB}{2} \sin \frac{mC}{2}$
according as m is of the form $4n+1$ or $4n+3$.
- $\cos A + \cos B + \cos C + \cos(A+B+C) = 4 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{B+C}{2} \right) \cos \left(\frac{C+A}{2} \right)$
- $\sin A + \sin B + \sin C - \sin(A+B+C) = 4 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{B+C}{2} \right) \sin \left(\frac{C+A}{2} \right)$

07 Trigonometrical Ratio of allied Angles

Equation of the type $a \cos \theta + b \sin \theta = c$

Case 1:

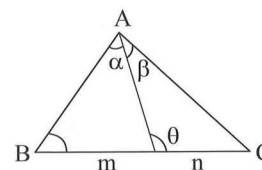
$$\text{if } c \leq \sqrt{a^2 + b^2} \Rightarrow \theta = \alpha + 2n\pi \pm \cos^{-1} \left(\frac{c}{\sqrt{a^2 + b^2}} \right)$$

Case 2:

$$\text{if } c > \sqrt{a^2 + b^2} \\ \text{No solution}$$

08 m-n Rule: in any triangle,

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta. = n \cot B - m \cot C.$$



09 Standard General Solutions of Trigonometrical Equations

- $\sin \theta = 0 \Leftrightarrow \theta = n\pi, \cos \theta = 0 \Leftrightarrow \theta = \left(2n\pi + \frac{\pi}{2} \right)$
- $\sin \theta = \sin \alpha$ and $\cos \theta = \cos \alpha \Leftrightarrow \theta = 2n\pi + \alpha, \eta \in \mathbb{Z}$
- $\sin^2 \theta = \sin^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha, \cos^2 \theta = \cos^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha$
- $\cos \theta = \cos \alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
- $\cos \theta = 1 \Leftrightarrow \theta = 2n\pi, \cos \theta = -1 \Leftrightarrow \theta = (2n+1)\pi$
- $\tan \theta = 0 \Leftrightarrow \theta = n\pi, \sin \theta = \sin \alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
- $\tan \theta = \tan \alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
- $\tan^2 \theta = \tan^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha, \sin \theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$

QUADRATIC EQUATION

1. Basic Results

The quantity ($D = b^2 - 4ac$) is known as the discriminant of the quadratic equation

- The quadratic equation has real and equal roots if and only if $D = 0$, i.e., $b^2 - 4ac = 0$.
- The quadratic equation has real and distinct roots if and only if $D > 0$, i.e., $b^2 - 4ac > 0$.
- The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$, i.e., $b^2 - 4ac < 0$.
- If $p + iq$ (p and q being real) is a root of the quadratic equation where $i = \sqrt{-1}$, then $p - iq$ is also a root of the quadratic equation.
- If $p + \sqrt{q}$ is an irrational root of the quadratic equation, then $p - \sqrt{q}$ is also a root of the quadratic equation provided that all the coefficients are rational
- The quadratic equation has rational roots if D is a perfect square and a, b, c are rational
- If $a = 1$ and b, c are integers and the roots of the quadratic equation are rational, then the roots must be integers.
- If the quadratic equation is satisfied by more than two numbers (real or complex), then it becomes an identity, i.e., $a = b = c = 0$.

3. Formation of an equation with given roots

A quadratic equation whose roots are

α and β is given by a

$$(x - \alpha)(x - \beta) = 0$$

$$\therefore x^2 - Sx + P = 0$$

$$\text{i.e., } x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\therefore x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Relation between the Roots of a Polynomial Equation of Degree n

Consider the equation

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$$

($a_0, a_1, \dots, a_n$ are real coefficients and $a_n \neq 0$).

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of equation (i). Then

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = a_n (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

Comparing the coefficients of like powers of x , we get

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_1 \alpha_4 + \dots + \alpha_2 \alpha_3 + \dots + \alpha_{n-1} \alpha_n = \frac{a_{n-2}}{a_n}$$

$$\alpha_1 \alpha_2 \dots \alpha_r + \dots + \alpha_{n-r+1} \alpha_{n-r+2} \dots \alpha_n = (-1)^r \frac{a_{n-r}}{a_n}$$

$$\alpha_1 \alpha_2 \dots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

2. Quick Look

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$
- $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = -\frac{b\sqrt{b^2 - 4ac}}{a^2}$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -\frac{b(b^2 - 3ac)}{a^3}$
- $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$
- $\alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2 = \left(\frac{b^2 - 2ac}{a^2}\right)^2 - 2\frac{c^2}{a^2}$
- $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = \frac{\pm b(b^2 - ac)\sqrt{b^2 - 4ac}}{a^4}$
- $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \frac{b^2 - ac}{a^2}$
- $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{b^2 - ac}{a^2}$
- $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = -\frac{bc}{a^2}$
- $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{a^2\beta^2}$

4. Equation in terms of the roots of another equations

If α, β are roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are

- $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (Replace x by $-x$)
- $\frac{1}{\alpha}, \frac{1}{\beta} \Rightarrow cx^2 + bx + a = 0$ (Replace x by $\frac{1}{x}$)
- $\alpha^n, \beta^n, n \in \mathbb{N} \Rightarrow \alpha\left(\frac{1}{x^n}\right) + b\left(\frac{1}{x^n}\right) + c = 0$ (Replace x by $\frac{1}{x^n}$)
- $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (Replace x by $\frac{x}{k}$)
- $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (Replace x by kx)
- $k + \alpha, k + \beta \Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (Replace x by $(x - k)$)
- $\alpha^{\frac{1}{n}}, \beta^{\frac{1}{n}}, n \in \mathbb{N} \Rightarrow \alpha(x^n)^2 + b(x^n)^2 + c = 0$ (Replace x by x^n)

5.

1. Condition for Two Quadratic Equations to have one Common Root

If $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α (say). Then $(dc - af)^2 = (bf - ce)(ae - bd)$.

2. Both roots are common

Then required condition is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

6. Quadratic Expression

The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x where a, b, c are real and $a \neq 0$. Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R} (a \neq 0)$. $f(x)$ can be rewritten as

$$f(x) = a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right] = a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right], \text{ where } D = b^2 - 4ac \text{ is discriminant}$$

of the quadratic expression. Then $y = f(x)$ represents a parabola whose axis is parallel to the

y -axis, with vertex at $A \left(-\frac{b}{2a}, -\frac{D}{4a} \right)$.

8. Maximum and minimum values of quadratic expression

Maximum and minimum value of quadratic expression can be found out by two methods:

1. Discriminate method:

In a quadratic expression $ax^2 + bx + c$.

2. Vertex of the parabola $Y = aX^2$ is $X = 0, Y = 0$.

$$\text{i.e., } x + \frac{b}{2a} = 0, y + \frac{D}{4a} = 0 \Rightarrow x = -\frac{b}{2a}, y = -\frac{D}{4a}$$

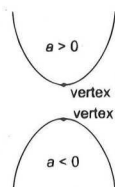
Hence, vertex of $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{D}{4a} \right)$.

• For $a > 0$, $f(x)$ has least value at $x = -\frac{b}{2a}$.

This least value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

• For $a < 0$, $f(x)$ has greatest value at $x = -\frac{b}{2a}$.

This greatest value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.



9. Interval in Which the Roots Lie

In some problems we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval.

For this we impose conditions on a, b and c . Let $f(x) = ax^2 + bx + c$.

1. If both the roots are positive, i.e., they lie in $(0, \infty)$, then the sum of the roots as well as the product of the roots must be positive.

$$\Rightarrow \alpha + \beta = -\frac{b}{a} > 0 \text{ and } \alpha\beta = \frac{c}{a} > 0 \text{ with } b^2 - 4ac \geq 0.$$

2. Similarly, if both the roots are negative, i.e., they lie in $(-\infty, 0)$ then the sum of the roots will be negative and the product of the roots must be positive, i.e., $\alpha + \beta = -\frac{b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$.

a. Both the roots are greater than a given number k if the following three conditions are satisfied:

$$D \geq 0, -\frac{b}{2a} < k \text{ and } a \cdot f(k) > 0.$$

b. Both the roots will lie in the given interval (k_1, k_2) if the following conditions are satisfied.

$$D \geq 0, k_1 < -\frac{b}{2a} < k_2 \text{ and } a \cdot f(k_1) > 0, a \cdot f(k_2) > 0.$$

c. Exactly one of the roots lies in the given interval (k_1, k_2) if $f(k_1) \cdot f(k_2) < 0$.

d. A given number k will lie between the roots if $a \cdot f(k) < 0$.

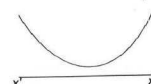
In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow a \cdot f(0) < 0$.

It also implies that the product of the roots is negative.

7. Sign of $f(x)$

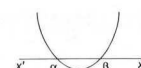
Depending on the sign of a and $b^2 - 4ac$, $f(x)$ may be positive, negative or zero. This gives rise to the following cases:

A $a > 0$ and $b^2 - 4ac < 0 \Leftrightarrow f(x) > 0 \forall x \in \mathbb{R}$.



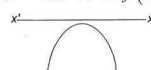
In this case the parabola always remains above the x -axis.

B $a > 0$ and $b^2 - 4ac > 0$. Let $f(x) = 0$ have two real roots α and β ($\alpha < \beta$).



Then $f(x) > 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$, and $f(x) < 0 \forall x \in (\alpha, \beta)$.

C $a < 0$ and $b^2 - 4ac < 0 \Leftrightarrow f(x) < 0 \forall x \in \mathbb{R}$.



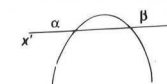
In this case the parabola always remains below the x -axis.

D $a < 0$ and $b^2 - 4ac = 0 \Leftrightarrow f(x) \leq 0 \forall x \in \mathbb{R}$.



In this case the parabola touches the x -axis and lies below the x -axis.

E $a < 0$ and $b^2 - 4ac > 0$



Let $f(x) = 0$ have two real roots α and β ($\alpha < \beta$). Then $f(x) < 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0 \forall x \in (\alpha, \beta)$.

COMPLEX NUMBER

1.

Iota

$$i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i \left[i^2 = -1, i^3 = -i, i^4 = 1 \right], i = -\frac{1}{i}$$

2.

Conjugate Complex Number

$$\bar{z} = a - ib \text{ for } z = a + ib$$

$$z + z' = 2a, z\bar{z} = a^2 + b^2.$$

• Properties

- (i) $(\bar{\bar{z}}) = z$ (ii) $z = \bar{z} \Leftrightarrow z$ is purely real
- (iii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary (iv) $\operatorname{Re}(z) = \operatorname{Re}(\bar{z}) = \frac{z + \bar{z}}{2}$
- (v) $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$ (vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- (vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- (ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
- (x) $z\bar{z}_2 + \bar{z}z_2 = 2\operatorname{Re}(\bar{z}z_2) = 2\operatorname{Re}(z\bar{z}_2)$
- (xi) $(z^n) = (\bar{z})^n$, (xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

3.

Modulus and Argument

$$z = a + ib$$

$$a = r \cos \theta, b = r \sin \theta$$

$$|z| = \sqrt{a^2 + b^2}, \text{ argument } \theta = \tan^{-1} \frac{b}{a}, 0 \leq \theta < 2\pi$$

principal values: $-\pi < \theta \leq \pi$

$$\text{1st quad } \theta = \tan^{-1} \frac{b}{a}$$

$$\text{2nd quad } \theta = \pi - \tan^{-1} \left(\frac{b}{|a|} \right)$$

$$\text{3rd quad } \theta = -\pi + \tan^{-1} \left(\frac{b}{a} \right)$$

$$\text{4th quad } \theta = -\tan^{-1} \left(\frac{|b|}{a} \right)$$

NOTE

- (i) $\operatorname{Arg}(0)$ is not defined.
- (ii) If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ & $\arg z_1 = \arg z_2$.
- (iii) If $\arg z = \pi/2$ or $-\pi/2$, z is purely imaginary.
- (iv) If $\arg z = 0$ or π , z is purely real.
- (v) Any two arguments of a complex number differ by $2n\pi$.

4.

Properties of Modulus

- (i) $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ & $|z| > 0$ if $z \neq 0$
- (ii) $-|z| \leq \operatorname{Re}(z) \leq |z|$ & $-|z| \leq \operatorname{Im}(z) \leq |z|$
- (iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- (iv) $z\bar{z} = |z|^2$
- (v) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$
- (vi) $|z_1 z_2| = |z_1| |z_2|$
- (vii) $|z^n| = |z|^n$
- (viii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$
- (ix) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$.
- (x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2|z_1| |z_2| \cos(\theta_1 - \theta_2)$
 $\theta_1 = \arg(z_1), \theta_2 = \arg(z_2)$.
- (xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

5.

Properties of argument

- (i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$
- (ii) $\arg z^n = n \arg z + 2k\pi$
- (iii) $\arg \frac{z_2}{z_1} = \theta \Rightarrow \arg \frac{z_1}{z_2} = 2k\pi - \theta, k \in \mathbb{I}$
- (iv) $\arg \bar{z} = -\arg z$

6.

De Moivre's Theorem

- (a) If n is any rational number, then
 $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

(b) If $z = r(\cos\theta + i\sin\theta)$ then

$$z^{1/n} = r^{1/n} \left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right) \right]$$

$$k = 0, 1, 2, \dots, (n-1)$$

7.

Square Root of a Complex Number

• Square roots of $z = a + ib$ are

$$\pm \left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}} \right] \text{ for } b > 0$$

$$\pm \left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}} \right] \text{ for } b < 0$$

• Square root of $i \rightarrow \pm \left(\frac{1+i}{\sqrt{2}} \right)$ • Square root of $-i \rightarrow \pm \left(\frac{1-i}{\sqrt{2}} \right)$

• Square root of $\omega \rightarrow \pm\omega^2$ • Square root of $\omega^2 \rightarrow \pm\omega$

8.

Cube roots of Unity

$$x = \sqrt[3]{1} = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2} = 1, \omega, \omega^2$$

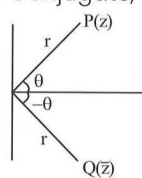
Properties

- (i) $1 + \omega + \omega^2 = 0$
- (ii) $\omega^3 = 1$
- (iii) $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$
- (iv) $\bar{\omega} = \omega^2$ & $(\bar{\omega})^2 = \omega$
- (v) Cube roots of unity lie on vertices of an equilateral triangle inscribed in a unit circle.
- (vi) $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$ if a, b, c are real
- (vii) $\omega^n + \omega^{n+1} + \omega^{n+2} = 0$

9.

Geometrical Meanings

1. Conjugate, $\bar{z}$:



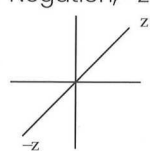
$$z = x + iy$$

$$\bar{z} = x - iy$$

$$z = (r, \theta)$$

$$\bar{z} = (r, -\theta)$$

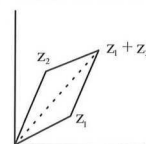
2. Negation, $-z$:



$$z = x + iy$$

$$-z = -x - iy$$

3. Sum of complex numbers:

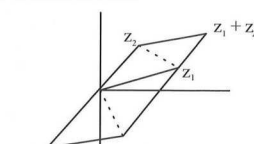


$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

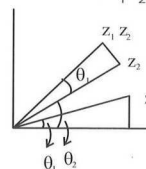
4. Difference:



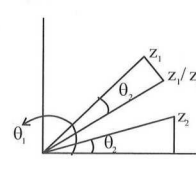
$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

5. Product: $z_1 z_2$



6. Quotient:



10.

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots + \infty$$

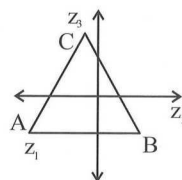
$$z = re^{i\theta} \Rightarrow \arg z = 2n\pi + \theta$$

$$\log z = \log r + i(\theta + 2n\pi)$$

∴ Logarithm of an imaginary number is not unique

11.

Concept of Rotation



$$AB = |z_2 - z_1|$$

$$BC = |z_3 - z_2|$$

$$CA = |z_1 - z_3|$$

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$$

$$\text{Also, } \left| \frac{z_3 - z_1}{z_2 - z_1} \right| = \frac{CA}{AB} \therefore \frac{z_3 - z_1}{z_2 - z_1} = \frac{CA}{AB} (\cos\alpha + i\sin\alpha)$$

If $z_1^2 + z_2^2 + z_3^2 + z_1z_2 + z_2z_3 + z_3z_1 = 0$, then the triangle ABC is equilateral.

LINEAR INEQUALITIES & LINEAR PROGRAMMING PROBLEMS

(1) Types of Inequalities

Numerical Inequalities: Relation between numbers e.g.: $3 < 5, 4 > 7$

Literal or Variable Inequalities : Relation between two variables or variable and numbers. e.g.: $y < 8$

Double Inequalities: Relationship from two sides e.g. $2 < x < 5$

Strict Inequalities: An inequality that uses the symbol $<$ or $>$ e.g.: $x < 5, 3 < 5$

Slack Inequality: An inequality that uses the symbols $\leq$ or $\geq$. e.g.: $y \leq 5$

Linear Inequalities in One Variable: An inequality which involves a linear function in one variable e.g.: $x < 5$

Linear Inequalities in two variables: An inequality which involves a linear function in two variables, e.g.: $3x + 2y < 5$

Quadratic Inequalities: An inequality which involves a quadratic function, eg.: $x^2 + 2x \leq 5$.

(2) Some important results

If a is a positive real number, then

$$|x| < a \Leftrightarrow -a < x < a \text{ i.e. } x \in (-a, a).$$

$$|x| \leq a \Leftrightarrow -a \leq x \leq a \text{ i.e. } x \in [-a, a].$$

$$|x| > a \Leftrightarrow x < -a \text{ or } x > a \quad |x| \geq a \Leftrightarrow x \leq -a \text{ or } x \geq a$$

$$|x| \geq a \Leftrightarrow x \leq -a \text{ or } x \geq a$$

Let r be a positive real number and a be a fixed real number. Then,

$$|x - a| < r \Leftrightarrow a - r < x < a + r \text{ i.e. } x \in (a - r, a + r).$$

$$|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r \text{ i.e. } x \in [a - r, a + r].$$

$$|x - a| > r \Leftrightarrow x < a - r \text{ or } x > a + r$$

$$|x - a| \geq r \Leftrightarrow x \leq a - r \text{ or } x \geq a + r$$

Let a and b be positive real numbers. Then

$$a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$$

$$a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$$

$$a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$$

$$a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$$

(3) Some Important Terms Related To Lpp

1. Feasible Region: The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as feasible region (or solution region). If we shade the region according to the given constraints, then the shaded area is the feasible region which is the common area of the regions drawn under the given constraints

2. Feasible Solution: Each point within & on the boundary of the feasible region represents feasible solution of constraints. Note that in the feasible region there are infinitely many points which satisfy the given condition.

3. Optimal Solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

(4) Mathematical Form Of Linear Programming Problems

The general mathematical form of a linear programming problem may be written as follow. **Objective Function:** $Z = C_1x + C_2y$ **Subject to constraints are:** $a_1x + b_1y \leq d_1, a_2x + b_2y \leq d_2$ etc and non-negative restrictions are $x \geq 0, y \geq 0$

(1) Objective Function: A linear function $Z = ax + by$, where a & b are constants, which has to be maximized or minimized according to a set of given conditions, is called a linear objective function.

(2) Decision Variables: In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.

(3) Constraints: The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0, y \geq 0$ are known as non-negative restrictions. In the constraints given in the general form of a LPP there may be any one of the 3 signs $\leq, =, \geq$

(5) Corner Point Method Of Solving Lpp

Steps Involved:

(1) Find the feasible region of the LPP & determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

(2) Evaluate the objective function $Z = ax + by$ at each corner point. Let M & m , respectively be the largest & smallest values of these points.

(3) (i) When the feasible region is bounded, M & m are the maximum & minimum values of Z . (ii) In case the feasible region is unbounded, we have:

(a) M is the maximum value of Z , if the open half plane determined by $ax + by > m$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z , if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

PERMUTATION & COMBINATION

01 Fundamental Principles of - Counting

(1) Multiplication Principle

If an operation can be performed in 'm' different way, following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations

(2) Addition Principle

If an operation can be performed in 'm' different ways & another operation, which is independent of the first operation, can be performed in 'n' different ways, then either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exclusive operations.

02 Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called permutations.

Factorial notation: $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$

$n! = n(n-1)!$ $0! = 1! = 1$

$2n! = 2n \times n!$ $[1, 3, 5, 7, \dots, (2n-1)]$

Factorials of negative integers are not defined.

03 Important results

01

Number of permutations of n different things, taking r at a time is denoted by ${}^n P_r$ or $P(n, r)$.

$${}^n P_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \dots (n-r+1)$$

Number of permutations of n different things taken all at a time = ${}^n P_n = n!$

$${}^n P_0 = 1, {}^n P_1 = n, {}^n P_n = n! \quad {}^n P_r = n({}^{n-1} P_{r-1}) = n(n-1)(n-2) \dots ({}^{n-2} P_{r-2}) \quad {}^{n-1} P_r = (n-r){}^{n-1} P_{r-1}$$

$$P_n = n! \quad {}^n P_r = {}^{n-1} P_r + r {}^{n-1} P_{r-1}$$

02 The number of permutation of n things taken all at a time, p are alike of one kind, q are alike of second kind r are alike of a third kind and $n = p + q + r$ is $\frac{n!}{p!q!r!}$

03 The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r .

04 Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement, is $r {}^{n-1} P_{r-1}$. Number of permutations of n different things, taken r at a time, when p particular things is to be always included in each arrangement, is $p! (r - (p - 1)) {}^{n-p} P_{r-p}$

05 Number of permutation of n different things taken r at a time, when a particular thing is never taken in each arrangement is ${}^{n-1} P_r$.

06 Number of permutations of n different things, taken all at a time, when m specified things always come together is $m! \times (n - m + 1)!$

07 Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m! \times (n - m + 1)!$

04 Circular Permutations

Arrangement round a circular table: Number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise & anticlockwise orders are taken as different.

Arrangement of beads around a circular necklace: Number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$ if clockwise & anticlockwise orders are taken as not different

Number of circular permutations of n different things taken r at a time is -

$$(i) \frac{{}^n P_r}{r}, \text{ when anti clockwise \& clockwise orders are taken as different.} \quad (ii) \frac{{}^n P_r}{2r}, \text{ when anticlockwise \& clockwise orders are not different.}$$

05 Combination

each of the different selections made by taking some or all at a time, irrespective of their arrangements, is called a combination.

The number of all combinations of n objects taken r at a time is denoted by $c(n, r)$ or ${}^n C_r$ or $\binom{n}{r}$

$${}^n C_r = \frac{n!}{r!(n-r)!} \quad {}^n C_r = \frac{n(n-1)(n-2) \dots (n-r+1)}{1.2.3 \dots r} \quad {}^n C_n = {}^n C_0 = 1 \quad {}^n C_r = \frac{{}^n P_r}{r!}$$

06 Number of Combinations without Repetition

The number of combination (selections or groups) that can be formed from n different objects taken r ($0 \leq r \leq n$) at a time is ${}^n C_r = \frac{n!}{r!(n-r)!}$

$$\bullet {}^n C_r \text{ is a natural number} \quad \bullet {}^n C_0 = {}^n C_n = 1, {}^n C_1 = n \quad \bullet {}^n C_r = {}^n C_{n-r} \quad \bullet {}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r \quad \bullet {}^n C_x = {}^n C_y \Leftrightarrow x = y \text{ or } x + y = n \quad \bullet n \cdot {}^{n-1} C_{r-1} = (n-r+1) {}^n C_{r-1}$$

$$\bullet \text{ If } n \text{ is even, then the greatest value of } {}^n C_r \text{ is } {}^n C_{n/2} \quad \bullet \text{ If } n \text{ is odd, then the greatest value of } {}^n C_r \text{ is } \frac{{}^n C_{n/2+1}}{2} \text{ or } \frac{{}^n C_{n/2-1}}{2} \quad \bullet {}^n C_r = \frac{n}{r} \cdot {}^{n-1} C_{r-1}$$

$$\bullet \frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r} \quad \bullet {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n \quad \bullet {}^n C_0 + {}^n C_2 + {}^n C_4 + \dots = {}^n C_1 + {}^n C_3 + {}^n C_5 + \dots = 2^{n-1} \quad \bullet 2n+1 {}^n C_0 + 2n+1 {}^n C_1 + 2n+1 {}^n C_2 + \dots + 2n+1 {}^n C_n = 2^{2n}$$

$$\bullet {}^n C_n + {}^{n+1} C_n + {}^{n+2} C_n + {}^{n+3} C_n + \dots + {}^{2n-1} C_n = 2^n {}^n C_{n+1}$$

07 Total number of divisors of a given natural number

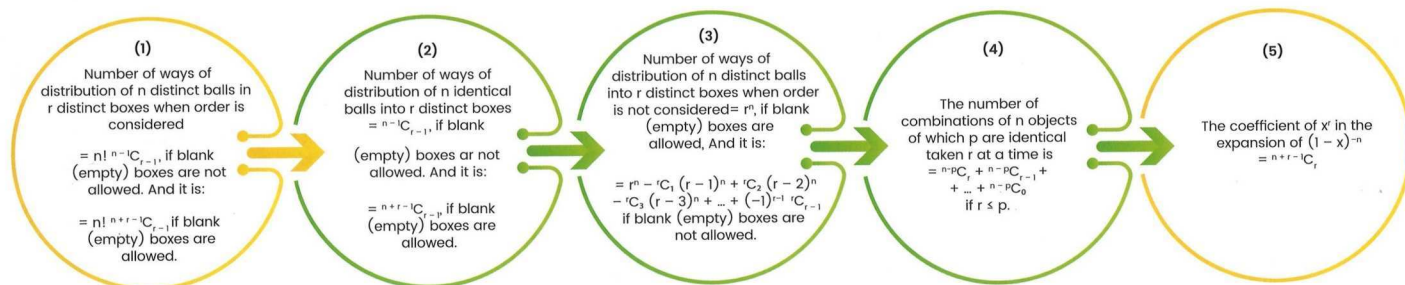
The number of factors of a given natural number greater than 1 we can write as, $N = P_1^{a_1} P_2^{a_2} P_3^{a_3} \dots P_n^{a_n}$ where $p_1, p_2, \dots, p_n$ are distinct prime numbers and $a_1, a_2, \dots, a_n$ are non-negative integers. $(a_1 + 1)(a_2 + 1) \dots (a_n + 1)$ ways. Sum of all the divisors of n is given by $\left(\frac{p_1^{a_1+1}-1}{p_1-1}\right) \cdot \left(\frac{p_2^{a_2+1}-1}{p_2-1}\right) \cdot \left(\frac{p_3^{a_3+1}-1}{p_3-1}\right) \dots \left(\frac{p_n^{a_n+1}-1}{p_n-1}\right)$

08 Derangements

Any change in the existing order of things is called a derangement. If 'n' things are arranged in a row, the number of ways in which they can be deranged so that none of them occupies its

original place is $n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right) = n! \sum_{r=0}^n (-1)^r \frac{1}{r!}$ And it is denoted by $D(n)$.

10 Distribution



11 Multinomial Theorem

If there are n_1 objects of one kind, n_2 objects of second kind and so on n_k objects of k th kind, then the number of ways of choosing r objects out of these objects is
 $=$ coeff of x^r in $(1+x+x^2+\dots+x^{n_1})(1+x+x^2+\dots+x^{n_2}) \dots (1+x+x^2+\dots+x^{n_k})$.

1

2

If one object of each kind is to be included in selection of (1), then the number of ways of choosing r objects is:
 $=$ coeff of x^r in $(x+x^2+\dots+x^{n_1})(x+x^2+\dots+x^{n_2}) \dots (x+x^2+\dots+x^{n_k})$

3

The number of possible arrangements permutations of p objects out of n_1 objects of kind 1, n_2 of kind 2 and so on is
 $= p!$ times the coefficient of x^p in the expansion

$$\left(1+x+\frac{x^2}{2!}+\dots+\frac{x^{n_1}}{n_1!}\right) \dots \left(1+x+\frac{x^2}{2!}+\dots+\frac{x^{n_k}}{n_k!}\right)$$

BINOMIAL THEOREM

01 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

$$(x + y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

$$\text{i.e., } (x + y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r \quad \dots (i)$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are called binomial coefficients and

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ For } 0 \leq r \leq n.$$

NOTE The binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots$ equidistant from beginning and end are equal, i.e., ${}^nC_r = {}^nC_{n-r}$.

03 SOME IMPORTANT EXPANSIONS

- (1) $(1 + x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$, i.e., $(1 + x)^n = \sum_{r=0}^n {}^nC_r x^r$
- (2) $(1 - x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 - \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n$
i.e., $(1 - x)^n = \sum_{r=0}^n (-1)^r {}^nC_r x^r$
- (3) $(x + y)^n + (x - y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \dots]$ and
 $(x + y)^n - (x - y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \dots]$
- (4) The coefficient of $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^n$ is nC_r .
- (5) If n is odd, then $(x + y)^n + (x - y)^n$ and $(x + y)^n - (x - y)^n$, both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$
- (6) If n is even, then $(x + y)^n + (x - y)^n$ has $\left(\frac{n+1}{2} + 1\right)$ terms and $(x + y)^n - (x - y)^n$ has $\frac{n}{2}$ terms.

05 APPLICATION OF GENERAL TERM

I. To Determine a Particular Term in the Expansion

In the expansion of $\left(x^a \pm \frac{1}{x^b}\right)^n$, if x^m occurs in T_{r+1} , then r is given by

$$n\alpha - r(\alpha + \beta) = m \Rightarrow r = \frac{n\alpha - m}{\alpha + \beta}$$

Thus in above expansion if constant term which is independent of

x , occurs in T_{r+1} , then r is determined by $n\alpha - r(\alpha + \beta) = 0 \Rightarrow r = \frac{n\alpha}{\alpha + \beta}$

07 Number of irrational terms in $(a^{1/p} + b^{1/q})^n \forall a, b \in \text{prime numbers}$

Method for finding terms free from radical or rational terms in the expansion of $(a^{1/p} + b^{1/q})^n \forall a, b \in \text{Prime numbers}$:

Find the general term $T_{r+1} = {}^nC_r (a^{1/p})^{n-r} (b^{1/q})^r = {}^nC_r a^{\frac{n-r}{p}} b^{\frac{r}{q}}$

Putting the values of $0 \leq r \leq N$, when indices of a and b are integers.

02 SOME USEFUL RELATIONS IN COMBINATORIAL

Some useful relations in combinatorial:

(1) ${}^nC_x = {}^nC_y \Rightarrow \text{either } x = y \text{ or } x + y = n$

$$(2) \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

$$(3) {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$(4) {}^nC_r + {}^{r+1}C_r + {}^{r+2}C_r + \dots + {}^nC_r = {}^{n+1}C_{r+1}$$

$$(5) {}^mC_r + {}^{m+1}C_r + {}^{m+2}C_r + \dots + {}^nC_r = {}^{n+1}C_{r+1} - {}^mC_{r+1}$$

$$(6) {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

$$(7) {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n {}^nC_n = 0$$

$$(8) {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \dots + {}^{2n+1}C_n = 2^{2n}$$

04 GENERAL TERMS

$T_{r+1} = {}^nC_r x^{n-r} y^r$ for $r = 0, 1, 2, 3, \dots, n$ defines respectively first, second, third, ..., n^{th} term.

- (1) In the binomial expansion of $(x + y)^n$, the p^{th} term from the end is $(n - p + 2)^{\text{th}}$ term from beginning.
- (2) The number of terms in the above binomial expansion are ${}^{n+1}C_1 = n + 1$.
- (3) The general term in the expansion of trinomial $(x + y + z)^n = \sum_{r,s,t} \frac{n!}{r!s!t!} x^r y^s z^t$ where $n \in \mathbb{N}$ and $r, s, t \in \{0, 1, 2, \dots, n\}$ and $r + s + t = n$. The number of terms in this expansion is ${}^{n+2}C_2$.
- (4) In the expansion of $(x + y)^n$, $n \in \mathbb{N}$, $T_r = \frac{{}^{n-r+1}C_r}{r} \cdot \frac{y}{x}$.
- (5) The coefficient of x^{n-1} in the expansion of $(x - 1)(x - 2) \dots (x - n) = -\frac{n(n+1)}{2}$.
- (6) The coefficient of x^{n-1} in the expansion of $(x + 1)(x + 2) \dots (x + n) = \frac{n(n+1)}{2}$.

06 MIDDLE TERM

The middle term depends upon the value of n .

- (1) When n is even, then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (odd). So there is only middle term, i.e., $\left(\frac{n+1}{2}\right)^{\text{th}}$ term.
- (2) When n is odd, then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (even). So there are two middle terms $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$ given by:

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \text{ And } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$$

- (1) When there are two middle terms in the expansion then their binomial coefficients are equal.
- (2) Binomial coefficient of middle term is the greatest binomial coefficient.
- (3) If n is even, then greatest binomial coefficient is ${}^nC_{\frac{n}{2}}$.
- (4) If n is odd, then greatest binomial coefficient are ${}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$.

NOTE

08 Numerically Greatest Term

Shortcut method: To find the greatest term (numerically) in the expansion of $(1+x)^n$.

- Calculate $m = \frac{|x|(n+1)}{|x|+1}$
- If m is integer, then T_m and T_{m+1} are equal and both are greatest term.
- If m is not integer, then $T_{[m]+1}$ is the greatest term, where $[.]$ denotes the greatest integral part.

10 MULTINOMIAL THEOREM (FOR POSITIVE INTEGRAL INDEX)

If n is positive integer and $a_1, a_2, a_3, \dots, a_m \in \mathbb{C}$, then

$$(a_1 + a_2 + a_3 + \dots + a_m)^n = \sum \frac{n!}{n_1! n_2! n_3! \dots n_m!} \dots a_1^{n_1} a_2^{n_2} a_m^{n_m}$$

Where $n_1, n_2, n_3, \dots, n_m$ are all non-negative integers subject to the condition, $n_1 + n_2 + n_3 + \dots + n_m = n$.

- The coefficients of $a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ in the expansion of $(a_1 + a_2 + a_3 + \dots + a_m)^n$ is
$$\frac{n!}{n_1! n_2! n_3! \dots n_m!}$$

- The greatest coefficients in the expansion of $(a_1 + a_2 + a_3 + \dots + a_m)^n$ is
$$\frac{(q!)^{m-r} [(q+1)!]^r}{n!}$$

Where q is the quotient and r is the remainder when n is divided by m .

- If n is +ve integer and $a_1, a_2, \dots, a_m \in \mathbb{C}$, then coefficients of x^r in the expansion of

$$(a_1 + a_2 x + \dots + a_m x^{m-1})^n \text{ is } \sum \frac{n! (a_1^{n_1} a_2^{n_2} \dots a_m^{n_m})}{n_1! n_2! n_3! \dots n_m!}$$

Where $n_1, n_2, \dots, n_m$ are all non-negative integers subject to the condition: $n_1 + n_2 + \dots + n_m = n$ and $n_2 + 2n_3 + 3n_4 + \dots + (m-1)n_m = r$.

- The number of distinct or dissimilar terms in the multinomial expansion $(a_1 + a_2 + a_3 + \dots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$

09 PROPERTIES OF BINOMIAL COEFFICIENTS

- In the expansion $(1+x)^n$, $2^n = C_0 + C_1 + C_2 + \dots + C_n$
- $0 = C_0 - C_1 + C_2 - C_3 + \dots$
- Sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1} .
 $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

11 BINOMIAL THEOREM FOR ANY INDEX

Statement

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!} x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} x^r + \dots \text{ terms up to } \infty$$

IMPORTANT TIPS

Expansion is valid only when $-1 < x < 1$.

nC_r cannot be used because it is defined only for natural number, so nC_r will be written as $\frac{n(n-1)\dots(n-r+1)}{r!}$

The number of terms in the series is infinite.

If first term is not 1, then make first term unity in the following way,
 $(x+y)^n = x^n \left[1 + \frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

12 GENERAL TERM

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

13 SOME IMPORTANT EXPANSIONS:

- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r + \dots$
- $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} (-x)^r + \dots$
- $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!} (x)^r + \dots$
- $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!} (-x)^r + \dots$

A sequence $\{a_n\}$ of real numbers is called an arithmetic progression (AP) if $a_{n+1} - a_n$ is constant for all positive integers $n \geq 1$, and this constant number is called the common difference of the AP.

1 Arithmetic progression (AP)

1 AP

2 General form of AP:

The terms of an AP with first term 'a' and common difference d are $a, a + d, a + 2d, a + 3d, \dots$, and the n^{th} term being $a_n + (n - 1)d$.

Quick Look

- If $\{a_n\}$ is an AP and k is any real number, then $\{a_n + k\}$ is also an AP with same common difference and $\{ka_n\}$ is also an AP.
- If $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then $\{a_n + b_n\}$ is also an AP.
- Product of two arithmetic progressions is also an AP if and only if one of them is a constant sequence
- Arithmetic means (AM's): If $a, A_1, A_2, \dots, A_n, b$ are in AP, then $A_1, A_2, \dots, A_n$ are called n AM's between a and b. The Kth mean A_k is given by $A_k = a + k \frac{(b-a)}{n+1}$ for $k = 1, 2, \dots, n$
- Sum to first n terms of an AP: Let S_n be the sum to first n terms of an AP with first term 'a' and common difference 'd'. Then $S_n = \frac{n}{2} [2a + (n - 1)d]$ or $S_n = \frac{n}{2} [\text{first term} + n^{\text{th}} \text{ term}]$
- If $A_1, A_2, \dots, A_n$ are n AM's between a and b then $A_1 + A_2 + \dots + A_n = \frac{n(a+b)}{2}$
- Ratio of n^{th} terms of two AP's: Let t_n be the n^{th} term of an AP whose first term is a and common difference d and S_n is its sum to first n terms. Let t'_n be the n^{th} term of another AP with first term b and common difference d whose sum of first n terms is S'_n . Then $\frac{t_n}{t'_n} = \frac{S_{2n-1}}{S'_{2n-1}}$
- Characterization of an AP: A sequence of real numbers is an arithmetic progression if and only if its sum of the first n terms is a quadratic expression in n with constant term zero.
- Helping points:
 - Three numbers in AP can be taken as $a - d, a, a + d$.
 - Four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$.
 - Five numbers in AP can be taken as $a - 2d, a - d, a, a + d, a + 2d$.

A sequence $\{a_n\}$ of non-zero real numbers is called GP if $a_n / a_{n-1} = a_{n+1} / a_n$ for $n \geq 2$. That is the ratio a_{n+1} / a_n is constant for $n \geq 1$ and this constant ratio is called the common ratio of the GP and is generally denoted by r.

1 Geometric progression (GP)

2 GP

GP with first term a, ar, ar².... whose n^{th} term is ar^{n-1}

2 General form

Quick Look

- If three numbers are in GP, then they can be taken as $a/r, a, ar$.
- If four numbers are in GP, then they can be taken as $a/r^3, a/r, ar, ar^3$.
- Sum to first n-terms of a GP: The sum of the first n-terms of a GP with first term 'a' and common ratio $r \neq 1$ is $\frac{a(1-r^n)}{1-r}$
- Sum to infinity of a GP: If $-1 < r < 1$ is the common ratio of a GP whose first term is a, then $S_\infty = a/1 - r$ is called sum to infinity of the GP.
- Geometric mean and geometric means: if three numbers a, b and c are in GP, then b is called the Geometric mean (GM) between a and c and $b^2 = ac$. If x and y are positive real numbers, then $x, \sqrt{xy}, y$ are in GP. If $a, g_1, g_2, \dots, g_n, b$ are in GP, then $g_1, g_2, \dots, g_n$ are called n geometric means between a and b.
- k^{th} GM g_k is given by $g_k = a(b/a)^{k/n+1}$ for $k = 1, 2, \dots, n$
- Product of n GM's between a and b is $(\sqrt[n]{ab})^n$

3

Arithmetic geometric progression (AGP):

Sequence of numbers of the form $a, (a + d)r, (a + 2d)r^2, \dots$ is called AGP

sum to n terms of an AGP is

$$\frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r}$$

and $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$ is the sum to infinity.

4

AM - GM inequality:

Let $a_1, a_2, \dots, a_n$ be positive reals. Then

$\frac{a_1 + a_2 + \dots + a_n}{n}$ is called AM of $a_1, a_2, \dots, a_n$ and

$(a_1 a_2 \dots a_n)^{1/n}$ is called their GM. Further

$\frac{a_1 + a_2 + \dots + a_n}{n} \geq (a_1 a_2 \dots a_n)^{1/n}$ and equality

holds if and only if $a_1 = a_2 = a_3 = \dots = a_n$

5. Harmonic progression (HP)

A sequence of non-zero reals is said to be in HP, if their reciprocals are in AP.

General form of an HP

Sequence of real numbers

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}, \dots$$

can be taken as general form of an HP.

Harmonic mean & Harmonic means:

- (1) If a, b, c are in HP, then b is called the Harmonic mean (HM) between a and c and in this case $b = \frac{2ac}{a+c}$.
- (2) If $a, h_1, h_2, \dots, h_n, b$ are in HP, then $h_1, h_2, \dots, h_n$ are called HM's between a and b further

$$h_k = \frac{ab(n+1)}{b(n+1) + k(a-b)} \text{ for } k = 1, 2, \dots, n$$

Theorem:

Let $a_1, a_2, \dots, a_n$ be positive real and A, G be AM and GM of the given numbers. Then,

$$H = \frac{n}{1/a_1 + 1/a_2 + \dots + 1/a_n}$$

which is called harmonic mean of $a_1, a_2, \dots, a_n$

Note: $A \geq G \geq H$ and equality holds if and only if $a_1 = a_2 = a_3 = \dots = a_n$

6. Some Useful Formulae

1. Telescopic Series: Suppose that we have to find the sum to n terms of a series $u_1 + u_2 + u_3 + \dots$.

$$u_k = a_k - a_{k+1} \text{ for all } k$$

$$\text{then } u_1 + u_2 + \dots + u_n = a_1 - a_{n+1}$$

For example, consider $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$ here, we have

$$\frac{1}{K(K+1)} = \frac{1}{K} - \frac{1}{K+1} \text{ for all } K \geq 2$$

2. Suppose that the term u_n of a given series is the product of r successive terms of an AP beginning with the n^{th} term of the AP; such that

$$u_n = [a + (n-1)d][a + nd] \dots [a + (n+r-2)d]$$

By choosing $a_n = u_n [a + (n+r-1)d]$, we can write

$$u_n = \frac{1}{(r+1)d} [a_n - a_{n-1}]$$

so that the sum to n terms is equal to $\frac{1}{(r+1)d} (a_n - a_0)$

For example, consider

$$(i) 1.2.3 + 2.3.4 + 3.4.5 + \dots$$

$$(ii) 1.3.5.7 + 3.5.7.9 + 5.7.9.11 + \dots$$

3. Suppose that the n^{th} term of a series is the reciprocal of the n^{th} term of the series given in II; that is,

$$u_n = \frac{1}{[a + (n-1)d][a + nd] \dots [a + (n+r-2)d]}$$

Then, we can choose $a_n = u_n [a + (n-1)d]$

so that $u_n = \frac{1}{(r-1)d} (a_{n-1} - a_n)$ and sum to n terms is given by

$$\frac{1}{(r-1)d} (a_0 - a_n)$$

For example, consider

$$(i) \frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \dots$$

$$(ii) \frac{1}{1.3.5.7} + \frac{1}{3.5.7.9} + \frac{1}{5.7.9.11} + \dots$$

STRAIGHT LINE

01 Slope of a Straight Line

If the line makes an angle θ with positive direction of x-axis, then $\tan \theta$ is called slope of the line and is denoted by m .

02 Various forms of Line

01 Slope intercept form: The line with slope m and y intercept c is $y = mx + c$

02 Slope point form: The line with slope m and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

03 Two point form: The line passing through the point (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

04 Intercept form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Here, a and b are x intercept and y intercept respectively which may be positive or negative

05 Normal form: The line whose normal makes an angle α with positive x axis and has length $= p$ is

$$x \cos \alpha + y \sin \alpha = p.$$

06 Distance or parametric form:

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

07 General form of line: The equation $ax + by + c = 0$ where a and b are not simultaneously zero is called general form of line.

Note

(a) x- intercept made by $ax + by + c = 0$ is $-\frac{c}{a}$.

(b) y- intercept made by $ax + by + c = 0$ is $-\frac{c}{b}$.

(c) Slope of the line $ax + by + c = 0$ is $-\frac{a}{b}$.

(d) Area of triangle which the line $ax + by + c = 0$ makes with coordinate axes $= \frac{c^2}{2ab}$.

03 Angle Between Two Lines

Let the slope of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are respectively m_1 and m_2 and if the angle between these lines be θ , then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \right|$$

(a) Condition for the lines to be parallel is

$$m_1 = m_2 \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

(b) Condition for the lines to be coincidental is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(c) Condition for the lines to be perpendicular is

$$m_1 m_2 = -1$$

$$a_1 a_2 + b_1 b_2 = 0$$

04 Family of Lines

01 Family of lines which are parallel to the line $ax + by + c = 0$ is $ax + by + \lambda = 0$ where $\lambda \in \mathbb{R}$

02 Family of lines which is perpendicular to the line $ax + by + c = 0$ is $bx - ay + \lambda = 0$ where $\lambda \in \mathbb{R}$

03 Family of lines passing through the intersection point of $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ is $L_1 + \lambda L_2 = 0$ where, $\lambda \in \mathbb{R}$

05 Distance between a Point and a line

Let (x_1, y_1) be the given point and $ax + by + c = 0$ be the given line then distance between them, is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

06 Distance between Two Parallel Lines

Let the equation of two parallel lines be $ax + by + c = 0$ and $ax + by + c' = 0$,

then distance between them is given by $P = \frac{|c - c'|}{\sqrt{a^2 + b^2}}$

Note

01 If the foot of perpendicular drawn from point (x_1, y_1) to the line $ax + by + c = 0$ be (h, k) , then,

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

02 If the image of point (x_1, y_1) in the line mirror $ax + by + c = 0$ be (α, β) , then $\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$

07 Concurrent Lines

Three or more lines are said to be concurrent if they have only one point in common. Let the three concurrent lines be $a_1x + b_1y + c_1 = 0$ where $r = 1, 2, 3$, then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Note

If lines are concurrent Δ must be zero but $\Delta = 0$ not necessarily imply the lines are concurrent.

08 Comparison of Two Points with Respect to a Line

Let the given line be $L(x, y) = ax + by + c = 0$ and the points are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then

1. If $L(x_1, y_1) \cdot L(x_2, y_2) > 0$ points P and Q lie on the same side of line $L = 0$

2. If $L(x_1, y_1) \cdot L(x_2, y_2) < 0$ points P and Q lie on the opposite side of line $L = 0$

09 Angle Bisectors of Angle Between Two Lines

The equations of angle bisectors of the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Note

In the above equation if c_1 and c_2 are of same sign, then taking the sign same as the sign of $a_1a_2 + b_1b_2$ we always get angle bisector of the given lines. Also by taking + in above formula we get the bisector of that angle region which contains origin.

1. Equation of straight line passing through given point (x_1, y_1) and making a given angle α with the given line

$$y = mx + c, \text{ are } y - y_1 = \frac{m - \tan \alpha}{1 + m \tan \alpha}(x - x_1) \text{ or } y - y_1 = \frac{m + \tan \alpha}{1 - m \tan \alpha}(x - x_1)$$

2. The image of the line $ax + by + c = 0$ in the line $X = \lambda$ is $a(2\lambda - x) + by + c = 0$

3. The image of the line $ax + by + c = 0$ in the line $y = \lambda$ is $ax + b(2\lambda - y) + c = 0$

10 Non-homogeneous equation of degree 2

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \dots$ (i) is called non-homogeneous equation of degree 2.

$$\text{Let, } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

1. Equation (i) represents a pair of straight lines if $\Delta = 0$, $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$.

2. If lines given by (i) have from $y = m_1x + c_1$ and $y = m_2x + c_2$, then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$, $|m_1 - m_2| = \frac{2\sqrt{h^2 - ab}}{a + b}$

3. If angle between the lines given by (i) be θ , then $\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$

Condition for line pair (i) to represents a pair of parallel lines is $h^2 = ab$

Condition for line pair (i) to represents a pair of perpendicular lines is $a + b = 0$

4. If the line pair given by (i) be the pair of parallel lines, then distance between them is $= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2 - bc}{b(a+c)}}$

5. Condition for the line pair (i) to represent coincidental lines is $h^2 = ab$ and $g^2 = ac$ and $f^2 = bc$

6. Point of intersection of the lines given by (i) is $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right) = (\alpha, \beta)$

7. Equation of angle bisector of the angle between line pair (i) is $\frac{(x - \alpha)^2 - (y - \beta)^2}{a - b} = \frac{(x - \alpha)(y - \beta)}{h}$

Note

In homogeneous case, $ax^2 + 2hxy + by^2 = 0$ replace g, f, c by 0.

11 Points to Remember

- 01 Equation of lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is given by $bx^2 - 2hxy + ay^2 = 0$

- 02 Two pair of straight lines viz. $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 = 0$ have

$$(a) \text{ a line in common if } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 4 \begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \begin{vmatrix} h_1 & b_1 \\ h_2 & b_2 \end{vmatrix}$$

$$(b) \text{ both lines in common if } \frac{a_1}{a_2} = \frac{h_1}{h_2} = \frac{b_1}{b_2}$$

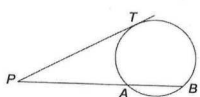
- 03 Equation of line pair joining the point of intersection of curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and line $lx + my = 1$ with origin is given by $ax^2 + 2hxy + by^2 + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0$

CIRCLE

1.

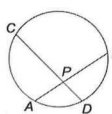
THINGS TO REMEMBER

1. Equal chords of a circle are always be at a equal distance from centre and vice-versa.
2. Equal chord subtends equal angle at the centre and vice-versa.
3. Angles inscribed by a chord at the same segment are always equal.
4. Central angle is always be the twice of the inscribed angle.
5. Perpendicular dropped from centre to the chord always bisect the chord.
6. Let length of perpendicular dropped from centre to chord is p and radius is r then length of chord is $2\sqrt{r^2 - p^2}$.
7. If PAB is a secant and PT is tangent, then



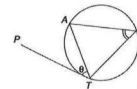
$$PA \times PB = PT^2$$

8. If two chords AB and CD intersect each other at point P, the



$$PA \times PB = PC \times PD$$

9. If PT is tangent and T is the point of contact, then if TA is chord such that $\angle ATP = \theta$, then



$$\angle AQT = \angle ATP$$

2.

EQUATION OF CIRCLE IN VARIOUS FORMS

1. Centre Radius Form : Equation of circle with centre (h, k) and radius r . $(x-h)^2 + (y-k)^2 = r^2$

2. Diameter form : Equation of circle with a diameter ends (x_1, y_1) and (x_2, y_2) are $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

3. General Form : $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $\Rightarrow (-g, -f)$

Radius $\Rightarrow \sqrt{g^2 + f^2 - c}$

Above equation represent circle only if, $g^2 + f^2 - c > 0$.

• If $g^2 + f^2 - c = 0$, then equation represents a point sometimes called a degenerate circle.

• If $g^2 + f^2 - c < 0$, then equation represents no locus. It is an imaginary equation.

3.

EQUATION OF CIRCLE IN PARTICULAR CASES

1. Equation of circle with centre (h, k) & passing through origin.

$$\Rightarrow (x-h)^2 + (y-k)^2 = h^2 + k^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky = 0$$

2. Equation of circle with centre (h, k) that touches x -axis

$$(x-h)^2 + (y-k)^2 = k^2$$

3. Equation circle with centre (h, k) that touches y -axis.

$$(x-h)^2 + (y-k)^2 = h^2$$

4. Equation of circle with radius r and which touches both the axes,

$$(x \pm r)^2 + (y \pm r)^2 = r^2.$$

4.

PARAMETRIC EQUATION

1. If equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, then its parametric form is

$$x + g = r \cos \theta; y + f = r \sin \theta; r = \sqrt{g^2 + f^2 - c};$$

θ = parameter

2. If equation of circle is $x^2 + y^2 = r^2$, then its parametric form is

$$x = r \cos \theta, y = r \sin \theta, \theta = \text{parameter}.$$

5.

FAMILY OF CIRCLE

1. Family of circle passing through the point of intersections of circle $S = 0$ and line $L = 0$ is $S + \lambda L = 0$.

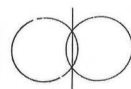
2. Equation of circle passing through two given points (x_1, y_1) (x_2, y_2) $S + \lambda L = 0$

Where $S = (x-x_1)(x-x_2) + (y-y_1)(y-y_2)$ and $L = 0$ is line passing through these points.

3. Equation of circle passing through the intersection of two circles $S = 0$ and $S' = 0$ is

Note

$S - S' = 0$ represents equation of common chord if circles intersect.



$$S + \lambda(S - S') = 0$$

4. Equation of the circle touching the given line given point (x_1, y_1) on it, $(x-x_1)^2 + (y-y_1)^2 + \lambda(ax+by+c) = 0$ at the

5. Equation of circumcircle of triangle whose sides are defined by the lines $L_1 = 0, L_2 = 0$ and $L_3 = 0$

$$L_1 L_2 + \lambda L_2 L_3 + \mu L_3 L_1 = 0$$

6. Equation of circle circumscribing the cyclic quadrilateral whose sides in order are given by lines $(L_1 = 0, L_2 = 0, L_3 = 0, L_4 = 0)$ $L_1 L_3 + \lambda L_2 L_4 = 0$

6.

INTERCEPT MADE BY CIRCLE ON COORDINATE AXES

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intersects x-axis at A and B and y-axis at C and D respectively, then :

1. Length of x-intercept, $AB = 2\sqrt{g^2 - c}$
2. Length of y-intercept, $CD = 2\sqrt{f^2 - c}$

Note

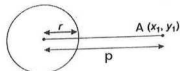
- Circle intersect x-axis at two distinct point only if $g^2 > c$. If $g^2 = c$, then circle touches x-axis. If $g^2 < c$, then circle neither intersect nor touches.
- If $f^2 > c$, then the circle intersect y-axis at two distinct points. If $f^2 = c$, then circle touches y-axis. If $f^2 < c$, neither touches nor intersect y-axis.
- Touches both axes simultaneously iff $g^2 = f^2 = c$

8.

DISTANCE BETWEEN A POINT AND CIRCLE

Let $S(x, y) = 0$ be the circle with radius r and $A(x_1, y_1)$ be the given point, then the distance between point A and circle S is equal to $|r - p|$

Where P is the distance between centre of circle and point P.



7.

POSITION OF POINT WITH RESPECT TO CIRCLE

Let circle is given by $S(x, y) = x^2 + y^2 + 2gx + 2fy + c = 0$ and point is (x_1, y_1) .

1. If $S(x_1, y_1) = 0$ point, lies on the circle.
2. $S_1 > 0$, point lies outside the circle
3. $S_1 < 0$, point lies inside the circle

9.

TANGENT TO A CIRCLE

- Equation of tangent at point (x_1, y_1) on $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$
- Condition for line $y = mx + c$ to be tangent to circle

$$x^2 + y^2 = a^2,$$

$$c^2 = a^2(l^2 + m^2)$$

And point at which it touches is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$

Note

- If line $Lx + my + c = 0$, touches the fixed circle with center (h, k) and radius r , then to find center and radius we can compare the given relation with

$$\frac{|ly + mk + c|}{\sqrt{l^2 + m^2}} = r$$

10.

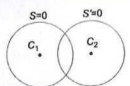
TWO CIRCLES CASES

Let $S = 0$ and $S' = 0$ be the circle with center C_1 and C_2 and radius r_1 and r_2 respectively. Let $C_1C_2 = d$.

INTERSECTING CIRCLE

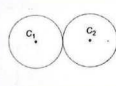
$$|r_1 - r_2| < d < r_1 + r_2$$

Number of common tangents = 2

**EXTERNALLY TOUCHING CIRCLE**

$$d = r_1 + r_2$$

Number of common tangents = 3

**ONE CIRCLE TOUCHES INTERNALLY THE OTHER**

$$d = |r_1 - r_2|$$

Number of common tangents = 1

**DISJOINT CIRCLES**

Have no point in common in their interior region and on the circles $d > r_1 + r_2$

Number of common tangents = 4

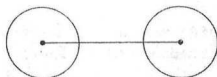
**ONE CIRCLE CONTAINED IN OTHER**

$$d < |r_1 - r_2|$$

Number of common tangents = 0

**NOTE**

- If the centre of both the circles lies on the same side of their common tangent, then such a tangent is said to be the direct common tangent.



- Length of direct common tangent

$$l = \sqrt{d^2 - (r_1 - r_2)^2}$$

- Length of indirect common tangent

$$l = \sqrt{d^2 - (r_1 + r_2)^2}$$

Locus of centre of circle which is touched by two noncongruent circles externally is always a hyperbola.

11.

CONCEPT**EQUATION**

1.	Equation of pair of tangents drawn from external point (x_1, y_1) to the curve $S = 0$.	$SS_1 = T^2$
2.	Chord of contact : From point (x_1, y_1) two tangents are drawn to curve $S = 0$, touching it at point A and B, then chord joining AB is called chord of contact of (x_1, y_1) with respect to curve $S = 0$.	$T = 0$
3.	Equation of tangent at point (x_1, y_1) on the curve $S = 0$.	$T = 0$
4.	If (x_1, y_1) be the internal point, then equation of chord of curve $S = 0$ with mid-point (x_1, y_1) .	$T = S_1$

12.

POLE AND POLAR

Locus of point of intersection of tangents drawn at the extremities of chords which are drawn from the fixed point (x_1, y_1) is called polar and for that polar, the point (x_1, y_1) is called pole.

Equation of polar at point (x_1, y_1) w.r.t curve $S = 0$ is given by,

$$T = 0$$

1. If point (x_1, y_1) i.e., pole is outside the curve, then chord of contact of it and polar to it has same equation.
2. If pole is lying on the curve, then tangent through it always coincide with polar of the pole.
3. If pole is inside the curve, then equation $T = 0$, represents only the polar.

PARTICULARLY FOR A CIRCLE

$$S(x, y) = x^2 + y^2 + 2gx + 2fy + c = 0 \text{ and } P(x_1, y_1)$$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c, \text{ then}$$

1. Length of tangent drawn from external point $P(x_1, y_1)$ is $L = \sqrt{S_1}$

2. Square of length of tangent i.e., S_1 is called power of external point $P(x_1, y_1)$ with respect to circle.

13.

THINGS TO REMEMBER

If from $P(x_1, y_1)$ two tangents are drawn to circle with radius R and length of tangent from P to the circle be L

1. Length of chord of contact of point

$$P(x_1, y_1) \\ AB = \frac{2RL}{\sqrt{R^2 + L^2}}$$

2. Area of $\triangle OAP = ar(\triangle OBP) = \frac{1}{2} RL$

3. Area of $\triangle OAB = \frac{R^3 L}{R^2 + L^2}$

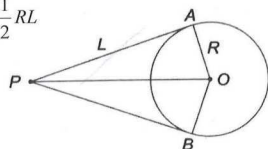
4. Area of $\triangle PAB = \frac{RL^3}{R^2 + L^2}$

5. Area of quadrilateral $PAOB = RL$

6. If angle between two pair of tangent is given by θ , then,

$$\begin{aligned} \tan\left(\frac{\theta}{2}\right) &= \frac{R}{L} \\ \Rightarrow \frac{\theta}{2} &= \tan^{-1}\left(\frac{R}{L}\right) \\ \Rightarrow \theta &= 2 \tan^{-1}\left(\frac{R}{L}\right) \end{aligned}$$

7. In above picture P, A, O and B are always be concyclic points and circle which passes through them has OP always as its diameter.



14.

DIRECTOR CIRCLE

- Locus of point of intersection of perpendicular tangents is called director circle.

- Equation of director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$

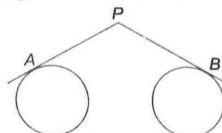
CIRCLE

- Director circle is always concentric with original circle.
- Ratio of radius of director circle to the radius of circle is $\sqrt{2}:1$

RADICAL AXIS OF TWO CIRCLE

The locus of a point from where tangents drawn to the given circle are equal in length is called radical axis of that pair of circles.

$$S - S' = 0$$



NOTE

- If circles are intersecting circles, radical axis of the circle has same equation as equation of common chord.
- If circles are touching circles radical axis coincides with their common tangents at the common point of contact.
- Radical axis is always perpendicular to line joining the centre of the circles.

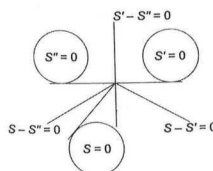
15.

RADICAL CENTRE

The point in the plane from where tangents drawn to three given circle are equal in length is called radical centre of those three circles.

$S = 0$, $S' = 0$ and $S'' = 0$ be the circles in the standard form.

Radical centre lies on the line $S - S' = 0$, $S' - S'' = 0$ and $S - S'' = 0$, i.e., radical centre always satisfy $S = S' = S''$



NOTE

If the point which satisfied of those three circles, $S = S' = S''$ is lie inside of any one of the circles, radical centres of the circles does not exist.

16. ANGLE OF INTERSECTION OF TWO CIRCLE

The angle between the tangents to the circle at the point of intersection of the curve is called angle of their intersection.

$$\cos \theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$$

17.

ORTHOGONAL CIRCLES

If angle of intersection of circles is 90° , then circles are said to be orthogonal circle : Condition for orthogonality of two circles is

$$r_1^2 + r_2^2 = d^2$$

• If circles are given by $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$, then above condition imply.

$$g_1g_2 + f_1f_2 = \frac{c_1 + c_2}{2}$$

Note

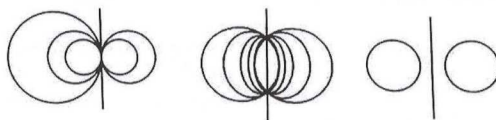
1. Taking centre on radical axis of two circles and radius equal to length of tangent from that point on either of circle. If circle being drawn, then angle of intersection of that circle with each of the given two circles is 90° .

2. Taking radical centre as a centre of circle and radius equal to the length of tangent from it to any of the three given circles if a circle being drawn, then that circle intersects all the three circles orthogonally.

18.

COAXIAL FAMILY OF CIRCLES

The family of circles in which each pair of circle has same radical axis is called co-axial family of circles.

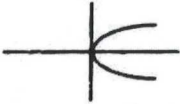
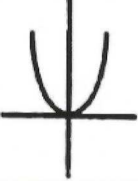


If two of the members of co-axial family are given by $S = 0$ and $S' = 0$, then co-axial family is described by

$$S + \lambda(S - S') = 0$$

PARABOLA

01 STANDARD PARABOLAS

S.NO	CONCEPT	$y^2 = 4ax (a > 0)$	$y^2 = -4ax (a > 0)$	$x^2 = 4ay (a > 0)$	$x^2 = -4ay (a > 0)$
1.	GRAPH				
2.	FOCUS	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
3.	DIRECTRIX EQUATION	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
4.	VERTEX	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
5.	LATUS RECTUM (L.R.)	$4a$	$4a$	$4a$	$4a$
6.	END OF L.R.	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
7.	AXIS	$y = 0$	$y = 0$	$x = 0$	$x = 0$
8.	FOCAL DISTANCE OF POINT (X,Y)	$(x + a)$	$(a - x)$	$(y + a)$	$(a - y)$
9.	PARAMETRIC EQUATION	$x = at^2, Y = 2at$	$x = at^2, Y = 2at$	$x = 2at, y = at^2$	$x = 2at, y = -at^2$
10.	PARAMETRIC POINT	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$

02 EQUAL OF PARABOLA IN VARIOUS CONDITIONS

1. Equation of parabola whose vertex is (α, β) and axis is parallel to x-axis is $(y - \beta)^2 = \pm 4a(x - \alpha)$ where $4a$ is L.R

2. Equation of parabola whose axis is parallel to x-axis is $x = ay^2 + by + c$

3. Equation of parabola whose vertex is (α, β) and axis is parallel to y-axis is $(x - \alpha)^2 = \pm 4a(y - \beta)$, $4a$ is L.R

4. Equation of parabola whose axis is parallel to y-axis is, $y = ax^2 + bx + c$

5. Equation of parabola whose axis is $ax + by + c = 0$ and tangent at vertex is $bx - ay + d = 0$

and whose latus rectum is $4A$ is $\left(\frac{ax + by + c}{\sqrt{a^2 + b^2}} \right)^2 = \pm 4A \frac{(bx - ay + d)}{\sqrt{a^2 + b^2}}$

Equation of Chord Joining Points t_1 and t_2 on Parabola $y^2 = 4ax$ $y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$

NOTE

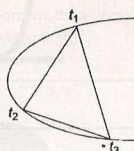
For the parabola $y^2 = 4ax$:

• Slope of the chord joining the points t_1 and $t_2 = \frac{2}{t_1 + t_2}$

- The chord joining points t_1 and t_2 if intersects axis at point $(k, 0)$, then $t_1 t_2 = -\frac{k}{a}$.
- Condition for the chord to be focal chord $t_1 t_2 = -1$
- If one end of focal chord is $(at^2, 2at)$, then coordinate of its other end will be obtained by replacing t by $-\frac{1}{t}$, i.e., the coordinate of the other end will be $\left(\frac{a}{t^2}, \frac{2a}{t}\right)$.
- Condition for the chord to subtend right angle at the vertex $t_1 t_2 = -4$.
- All the chord which subtends right angle at the vertex passes through fixed point $(4a, 0)$.
- Focal chord never subtend right angle at vertex or line subtending right angle at vertex never be a focal chord.
- Let the variable chord is drawn from point $P(2a, 0)$, meets parabola at Q and R then

$$\frac{1}{PQ^2} + \frac{1}{PR^2} = \frac{1}{4a^2} \text{ (constant)}$$

• Area of triangle whose vertices are given by $A(t_1)$, $B(t_2)$ and $C(t_3)$ on parabola



$$\Delta = \frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

03 EQUATION OF TANGENT & NORMAL

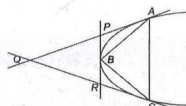
- Equation of Tangent at Point (x_1, y_1) on Parabola $y^2 = 4ax$ $T = 0 \Rightarrow yy_1 = 2a(x + x_1)$
- Equation of normal at 't', $tx + y = 2at + at^3$
- Equation of tangent at point t, $ty = x + at^2$
- Equation of normal with slope m, $y = mx - 2am - am^3$
- Point of normality, $(am^2, -2am)$
- Equation of tangent with slope m
- has point of contact $\frac{a}{m^2}, \frac{2a}{m}$
- $y = mx + \frac{a}{m}$

04 PROPERTIES OF TANGENT

- Condition for line with finite slope say $y = mx + c$ to be tangent to parabola is $c = \frac{a}{m}$. Hence $m \neq 0$, then, To be a chord $cm < a$. Neither chord nor tangent is $cm > a$.
- Image of the focus in any tangent always lies on directrix.
- At any point P if tangent is drawn at parabola meet the axis at T and normal to parabola meet axis on N then $ST = SN = SP$ where S is focus.
- i) Tangent at points t_1 and t_2 always intersect on $[at_1 t_2, a(t_1 + t_2)]$.

ii) Area of triangle formed by tangents drawn on points $(t_1, t_2 \text{ and } t_3)$ on parabola.

Let $A(t_1)$, $B(t_2)$ and $C(t_3)$ be the points on parabola an tangent at A, B and C forming ΔPQR .



$$\Delta PQR = \frac{1}{2} \Delta ABC$$

iii) Tangents if drawn by taking any point on its directrix always touches the parabola at the ends of a focal chord.

iv) If tangents are drawn from P touching the parabola at A and B. Then AB is called chord of contact of P with respect to parabola if $P(h, k)$, then

$$\text{Length of } AB = \frac{\sqrt{S_1} \sqrt{k^2 + 4a}}{a}; \text{ where } S_1 = k^2 - 4ah \quad \bullet \quad \Delta PAB = \frac{(S_1)^{\frac{3}{2}}}{2a}$$

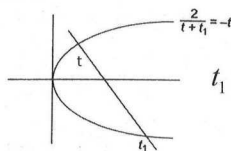
v) If tangents are drawn at any 3 points on the parabola, then orthocentre of triangle formed by them is always lie on directrix.

05 PROPERTIES OF NORMAL

For the parabola $y^2 = 4ax$

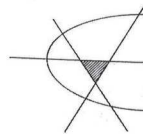
- i) Line $y = mx + c$ is a normal to the parabola if $c = -2am - am^3$
- ii) From a point there can be maximum 3 normals to the parabola.
- iii) If from point (h, k) , three normals be drawn with slope $m_1 + m_2 + m_3 = 0$, m_1 , m_2 and m_3 , then and point of normalities corresponding to these are given by $(am_1^2 - 2a, m_1)$, $(am_2^2 - 2a, m_2)$ and $(am_3^2 - 2a, m_3)$, called conormal points.
- iv) If from a point 3 normals are possible, then sum of the ordinates of conormal points is always equal to zero.

- v) If a normal is drawn at point t on parabola meets the parabola again at point t_1 then t_1 is always equal to



$$t_1 = -t - \frac{2}{t}$$

- viii) Area of triangle formed by the normal drawn at points t_1, t_2 and t_3



$$\Delta = \frac{1}{2} \left| a^2 (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3) \right|$$

- ix) From a point $(h, 0)$ three normals are possible on the parabola if $h > 2a$, i.e. if $h \leq 2a$, then from $(h, 0)$ there is only one normal, i.e. axis of parabola.

- x) The length of subnormal at any point on parabola $y^2 = 4ax$ is constant known as semi latus rectum.

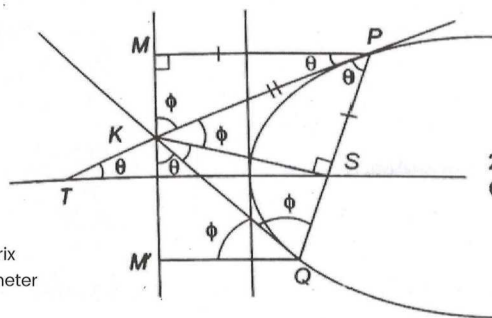
06 OPTICAL PROPERTY OF PARABOLA

Let $y^2 = 4ax$ be a parabola then all the rays from $X \rightarrow \infty$ parallel the axis of parabola after reflecting from internal surface of parabolic mirror always passes through focus of parabola.

Features of Parabola

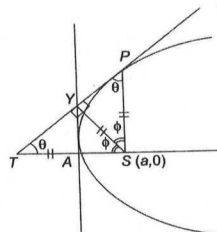
Let P be any point on the parabola and tangent at P meets the axis at T and directrix at K . Let tangent from K meets parabola at P and Q . Let S be the focus of parabola, PM is perpendicular to directrix and QM is also perpendicular to directrix.

- $\angle TPM = \angle TPS = \angle STP = \angle SKQ$
- Tangents at the ends of focal chord always meet each other on directrix at an angle of 90° . Therefore, mutually orthogonal tangents touches the parabola at the end of a focal chord. Since tangent from directrix are mutually perpendicular. Hence, directrix is called director circle of parabola.
- $\therefore \angle PKQ = 90^\circ$
- Taking PQ , i.e., focal chord as a diameter if a circle being drawn that always touches the directrix at the point where tangent at point P & Q on the parabola meet the directrix.
- In the above figure $\triangle PMK \cong \triangle PSK$ and $\triangle SKQ \cong \triangle MKQ$
 $\therefore \angle KSP = 90^\circ$, i.e., the intercept of tangent between point of contact and directrix always subtend right angle at the focus, therefore taking that intercept as a diameter if a circle being drawn that always passes through focus of parabola.
- If tangent being drawn at point P on parabola meets tangent at vertex at Y .
- If A be the vertex. $\Rightarrow \frac{SY}{SP} = \frac{AS}{SY} \Rightarrow SY^2 = AS \cdot SP$. Hence, AS, SY, SP are always in G.P.
- $\triangle SPY \cong \triangle STY$
- Perpendicular drawn from focus to the tangent always bisects the angle between axis and focal radii of point of contact.



$$2\theta + 2\phi = 180^\circ$$

$$\theta + \phi = 90^\circ$$



07 CONGRUENT OR EQUAL PARABOLA

- Two parabolas are said to be equal or congruent if they have the same latus rectum

- Two equal parabolas with the same axis different vertex if sketch in the same direction never meet each other.

Length of focal chord, $l = \left| a(t_1 - t_2)^2 \right|$ $l = \left| a \left(t_1 + \frac{1}{t_1} \right)^2 \right|$

08 DIAMETER OF A PARABOLA

$\Rightarrow$ Latus of mid-point of the set of parallel chords is called diameter let $y^2 = 4ax$ be the parabola and chords are drawn with slope m , then equation of corresponding diameter will be, $y = \frac{2a}{m}$

- Each and every line parallel to axis of parabola is its diameter.

- Tangents at the end of the chord always meet each other on the diameter corresponding to the chord.

Ellipse

1.

Ellipse

The condition for second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent an ellipse is that $h^2 - ab < 0$ and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

Standard Equation of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2)$$

Since $e < 1$, therefore $a^2(1 - e^2) < a^2 \Rightarrow b^2 < a^2$

2.

Ellipse— Basic fundamentals

Equation	$\left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$	
	For $a > b$	For $b > a$
Centre	(0, 0)	(0, 0)
Vertices	($\pm a$, 0)	(0, $\pm b$)
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	($\pm a$, 0)	(0, $\pm b$)
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a , b and e	$b^2 = a^2(1 - e^2)$	$a^2 = b^2(1 - e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm \frac{b^2}{a} \right)$	$\left(\pm \frac{a^2}{b}, \pm be \right)$
Parametric equations	$(a \cos \phi, b \sin \phi)$	$(a \cos \phi, b \sin \phi) \quad (0 \leq \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = b - ey_1$ and $S'P = b + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$2a/e$	$2b/e$
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

3.

Parametric form of the Ellipse

For the equation of ellipse in standard form the parametric form will be given by taking $x = a \cos \phi$, $y = b \sin \phi$, where ϕ is the eccentric angle, in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

4.

Special forms of an Ellipse

1. If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its

equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

If we shift the origin at (h, k) without rotating the coordinate axes, then $x = X + h$ and $y = Y + k$.

2. If the equation of the curve is $\frac{(lx + my + n)^2}{a^2} + \frac{(mx - ly + p)^2}{b^2} = 1$

where $lx + my + n = 0$ and $mx - ly + p = 0$ are perpendicular lines, then we substitute $\frac{lx + my + n}{\sqrt{l^2 + m^2}} = X$, $\frac{mx - ly + p}{\sqrt{l^2 + m^2}} = Y$, to put the equation in the standard form.

5.

Position of a Point with Respect to an Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of an ellipse. The point lies outside, on or inside the ellipse as if

$$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0, < 0.$$

6.

Auxiliary Circle

The circle described on the major of axis of an ellipse as diameter is called an auxiliary circle of the ellipse.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, then its auxiliary circle is $a^2 + y^2 = a^2$.

7.

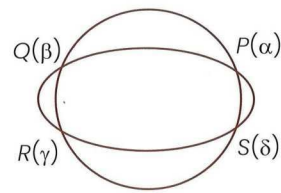
Equation of the Chord Joining Two Points on an Ellipse

The equation of the chord joining two points having eccentric angles θ and ϕ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a} \cos \left(\frac{\theta + \phi}{2} \right) + \frac{y}{b} \sin \left(\frac{\theta + \phi}{2} \right) = \cos \left(\frac{\theta - \phi}{2} \right).$$

Concyclic Points

If circle intersects an ellipse in four points. They are called concyclic points and the sum of their eccentric angles is an even multiple π .



If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points P, Q, R, S of circle intersect an ellipse then, $\alpha + \beta + \gamma + \delta = 2n\pi$, where n is any integer.

8.

Equations of Tangent in Different Forms

1. **Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Note: Point of intersection of tangent drawn at point θ and ϕ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\left(\frac{a \cos \left(\frac{\theta + \phi}{2} \right)}{\cos \left(\frac{\theta - \phi}{2} \right)}, \frac{b \sin \left(\frac{\theta + \phi}{2} \right)}{\cos \left(\frac{\theta - \phi}{2} \right)} \right)$$

2. **Slope form:** If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then $c^2 = a^2 m^2 + b^2$. Hence, the straight line $y = mx \pm \sqrt{a^2 m^2 + b^2}$ always represents the tangents to ellipse.

Points of contact: Line $y = mx \pm \sqrt{a^2 m^2 + b^2}$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } \left(\frac{-a^2 m}{c}, \frac{b^2}{c} \right) \text{ where } c^2 = a^2 m^2 + b^2$$

3. **Parametric form:** The equation of tangent at any point $\phi(a \cos \phi, b \sin \phi)$ is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$.

9.

Pair of Tangents

Equation of Pair of Tangents $SS_1 = T^2$

Pair of tangents: Let $P(x_1, y_1)$ be any point lying outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let a pair of tangents PA, PB can be drawn to it from P .

Then the equation of pair of tangents PA and PB is $SS_1 = T^2$ Where

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1, T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

10.

Director Circle

The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.

Equation of director circle is

$$x^2 + y^2 = a^2 + b^2,$$

It is a circle concentric with ellipse and has radius $\sqrt{a^2 + b^2}$.

11.

Equations of Normal in Different Forms

1. **Point form:** The equation of the normal at (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

2. **Parametric form:** The equation of the normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a \cos \phi, b \sin \phi) \text{ is } a x \sec \phi - b y \csc \phi = a^2 - b^2.$$

3. **Slope form:** If m is the slope of the normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then the equation of normal is}$$

$$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$$

12.

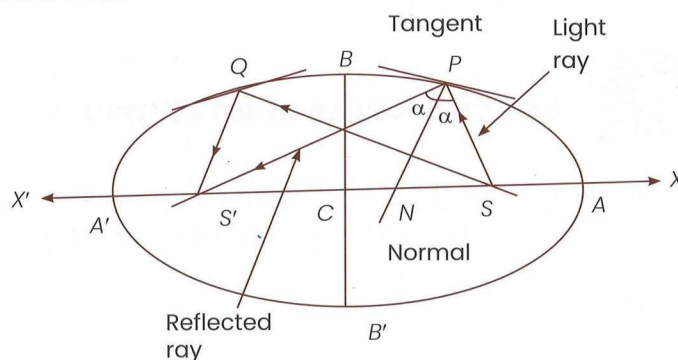
Properties of Eccentric Angles of the Co-normal Points

- The sum of the eccentric angles of the co-normal points on the ellipse of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to odd multiple of π .
- If α, β, γ are the eccentric angles of three points on the ellipse, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

13.

Reflection Property of an Ellipse

Let S and S' be the foci and PN the normal at the point P of the ellipse, then $\angle SPS' = \angle SQS'$. Hence, if an incoming light ray aimed towards one focus strike the concave side of the mirror in the shape of an ellipse, then it will be reflected towards the other focus.



14.

Chord of Contact

If PQ and PR be the tangents through point $P(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then the equation of the chord of contact } QR \text{ is}$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \text{ or } T = 0 \text{ at } (x_1, y_1).$$

Equation of Chord with Mid-Point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose mid-point be (x_1, y_1) is $T = S_1$, where

$$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0, \quad S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0.$$

HYPERBOLA

01

Definition of Hyperbola

A hyperbola is the particular case of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

When, $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

i.e., $\Delta \neq 0$ and $h^2 > ab$

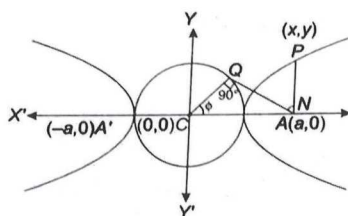
03

Auxiliary Circle of Hyperbola

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis A'A. Therefore, circle drawn with

centre C and segment A'A as a diameter is called auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of the auxiliary circle is $x^2 + y^2 = a^2$.



Let $\angle QCN = \phi$

Here, P and Q are the corresponding points on the hyperbola and the auxiliary circle ($0 \leq \phi < 2\pi$).

Note : Here ϕ is called eccentric angle of point P.

04

Parametric equations of hyperbola

The equations $x = a \sec \phi$ and $y = b \tan \phi$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

This $(a \sec \phi, b \tan \phi)$ lies on the hyperbola for all values of ϕ .

Position of points Q on auxiliary circle and the corresponding point P which describes the hyperbola and $0 \leq \phi < 2\pi$

ϕ varies from	Q($a \cos \phi, b \sin \phi$)	P($a \sec \phi, b \tan \phi$)
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ to 2π	IV	IV

02

Hyperbola

Hyperbola Fundamentals	Hyperbola	CONJUGATE HYPERBOLA
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0,0)	(0,0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	($\pm ae, 0$)	(0, $\pm be$)
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric co-ordinates	($a \sec \phi, b \tan \phi$), $0 \leq \phi < 2\pi$	($b \sec \phi, a \tan \phi$), $0 \leq \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii ($S'P - SP$)	2a	2b
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

- If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.
- The foci of a hyperbola and its conjugate are concyclic.

05

Position of a Point with Respect to a Hyperbola

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, $P(x_1, y_1)$ will lie inside, on or outside the hyperbola

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

06

Equation of the Chord joining Two Points on the Hyperbola

The equation of the chord joining the points $P(a \sec \phi_1, b \tan \phi_1)$ and $(a \sec \phi_2, b \tan \phi_2)$

$$y - b \tan \phi_1 = \frac{b \tan \phi_2 - b \tan \phi_1}{a \sec \phi_2 - a \sec \phi_1} (x - a \sec \phi_1) \quad \frac{x}{a} \cos \left(\frac{\phi_1 - \phi_2}{2} \right) - \frac{y}{b} \sin \left(\frac{\phi_1 + \phi_2}{2} \right) = \cos \left(\frac{\phi_1 + \phi_2}{2} \right)$$

Note

- If the chord joining two points $(a \sec \phi_1, b \tan \phi_1)$ and $(a \sec \phi_2, b \tan \phi_2)$ passes through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$.

07

Intersection of a Line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points may be real, coincident or imaginary according as $c^2 >, = < a^2 m^2 - b^2$.

08

Condition for tangency

If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2 m^2 - b^2$.

09

Equations of Tangent in Different Forms

01

Point form: The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1) \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

02

Parametric form: The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \phi, b \tan \phi)$ is $\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1$

Note

Point of intersection of tangents drawn at point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{a \cos \left(\frac{\theta - \phi}{2} \right)}{\cos \left(\frac{\theta + \phi}{2} \right)}, \frac{b \sin \left(\frac{\theta + \phi}{2} \right)}{\cos \left(\frac{\theta + \phi}{2} \right)} \right)$$

03

Slope form: The equations of tangents of slope m to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ are } y = mx \pm \sqrt{a^2 m^2 - b^2} \text{ and the co-ordinates of points}$$

of contacts are $\left(-\frac{a^2 m}{c}, -\frac{b^2}{c} \right)$ where $c^2 = a^2 m^2 - b^2$.

Clearly for the existence of tangent with slope m to the hyperbola $|m| > \frac{b}{a}$ (where $a, b > 0$).

Note

If the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2 l^2 - b^2 m^2 = n^2$.

If the straight line $x \cos \alpha + y \sin \alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$. • Two tangents can be drawn from an outside point to a hyperbola.

10

Equation of Pair of Tangents

If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then a pair of tangents PQ, PR can be drawn to it from P

The equation of pair of tangents PQ and PR is $SS_1 = T^2$, where,

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1, S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1, T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

Director circle: The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

11

Equations of Normal in Different Forms

01

Point form : The equation of normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1) \text{ is}$$

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2.$$

02

Parametric form : The equation of normal at $(a \sec \theta, b \tan \theta)$, to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$ax \cos \theta + by \cot \theta = a^2 + b^2.$$

03

Slope form : The equation of the normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ in terms of the slope } m \text{ of}$$

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$$

04

Condition of normality : If

$$y = mx + c \text{ is the normal of}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ then } c = \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - m^2 b^2}} \text{ or}$$

$$c = \frac{m^2(a^2 + b^2)^2}{(a^2 - m^2 b^2)^2}, \text{ which is condition of normality.}$$

12

Important Tips

01

In general, four normals can be drawn to a hyperbola from any point and if $\alpha, \beta, \gamma, \delta$ be the eccentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ is an odd multiple of π .

02

if α, β, γ are the eccentric angles of three points on the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ at which the normals are concurrent, then}$$

$$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0.$$

03

If the normal at P meets the transverse axis in G, then $SG = e \cdot SP$.

Also the tangent and normal bisect the angle between the focal distances of P

04

If the normal at P meets the transverse axis in G and conjugate axis at g, then

$$PG : Pg = b^2 : a^2.$$

13

Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola

Let PQ and PR be tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ drawn from any external point $P(x_1, y_1)$.

Then equation of chord of contact QR is or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$ (At x_1, y_1)

14

Equation of the Chord of the Hyperbola whose Mid-point (x_1, y_1) is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the given point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1, \text{ i.e., } T = S_1.$$

Note

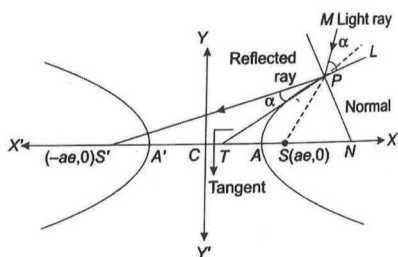
The length of chord cut off by hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the line

$$y = mx + c \text{ is } \frac{2ab\sqrt{[c^2 - (a^2 m^2 - b^2)](1+m^2)}}{(b^2 - a^2 m^2)}.$$

15

Reflection Property of the Hyperbola

If an incoming light ray passing through one focus (S) strike convex side of the hyperbola, then it will get reflected towards other focus (S'). $\angle TPS' = \angle LPM = \alpha$



Note

Hyperbola and ellipse are called orthogonal curves to each other iff they are confocal (i.e., they have the same foci).

16

Asymptotes of a Hyperbola

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$y = \pm \frac{b}{a}x \text{ or } \frac{x}{a} \pm \frac{y}{b} = 0.$$

Important Tips

The product of length of perpendiculars drawn from any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ to the asymptotes is } \frac{a^2 b^2}{a^2 + b^2}.$$

The tangent at any point P on hyperbola if meet its asymptotes at Q and R, then :

- (i) The midpoint of QR is always P, (ii) Area of triangle QCR is always " ab " where C is the of hyperbola and $2a =$ length of transverse axis, $2b =$ length of conjugate axis of hyperbola.

17

Rectangular or Equilateral Hyperbola

(i) **Definition** : A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$. The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and coefficient of $x^2 +$ coefficient of $y^2 = 0$.

18

Parametric coordinates of a point on the hyperbola $xy = c^2$

If t is non-zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $\left(ct, \frac{c}{t}\right)$. The point

$\left(ct, \frac{c}{t}\right)$ on the hyperbola $xy = c^2$ is generally referred as the point ' t '.

For rectangular hyperbola the coordinates of foci are $(\pm a\sqrt{2}, 0)$ and directrices are $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

19

Equation of the chord joining points t_1 and t_2

The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the

$$\text{hyperbola } xy = c^2 \text{ is } y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1)$$

$$\Rightarrow x + yt_1t_2 = c(t_1 + t_2).$$

20

Equation of tangent in different forms

(i) **Point form** : The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is

$$xy_1 + yx_1 = 2c^2 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2.$$

(ii) **Parametric form** : The equation of the

tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is

$$\frac{x}{t} + yt = 2c \text{ On replacing } x_1 \text{ by } ct \text{ and } y_1 \text{ by } \frac{c}{t}$$

on the equation of the tangent at (x_1, y_1) ,

$$\text{i.e., } xy_1 + yx_1 = 2c^2, \text{ we get } \frac{x}{t} + yt = 2c.$$

Note

$$\text{Point of intersection of tangents at } t_1 \text{ and } t_2 \text{ is } \left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right).$$

22

Intersection of a Circle and a Rectangular Hyperbola

If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and

the parameters of these four points t_1, t_2, t_3 and t_4 respectively, then :

21

Equation of the chord joining points t_1 and t_2

(i) **Point form** : The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(ii) **Parametric form** : The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

Note

• The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in t . So, in general,

• If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in t' then $t' = -\frac{1}{t^3}$.

• Point of intersection of normals at t_1 and t_2 is

$$\left(\frac{c\{t_1t_2(t_1^2+t_1t_2+t_2^2)-1\}}{t_1t_2(t_1+t_2)}, \frac{c\{t_1^3t_2^3(t_1^2+t_1t_2+t_2^2)\}}{t_1t_2(t_1+t_2)}\right).$$

• A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola

• All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.

01

(i)
 $\sum t_i = -\frac{2g}{c}$

(ii)
 $\sum t_1t_2 = \frac{k}{c^2}$

(iii)
 $\sum t_1t_2t_3 = -\frac{2f}{c}$

(iv)
 $t_1t_2t_3t_4 = 1$

(v)
 $\sum \frac{1}{t_i} = -\frac{2f}{c}$

STATISTICS

1.

Measures of Dispersion

The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four dispersions are: Range, Mean deviation, Standard deviation and Variance.

2.

Variance & standard Deviation of a discrete frequency distribution

Variance of a discrete frequency distribution

$$\text{Var}(\sigma^2) = \frac{1}{N} \sum f_i (x_i - \bar{x})^2$$

Standard Deviation (S.D.) of a discrete frequency distribution

$$\text{S.D}(\sigma) = \sqrt{\frac{1}{N} \sum f_i (x_i - \bar{x})^2}$$

where $N = \sum f_i$ and $\bar{x}$ = mean

3.

Variance and Standard Deviation for ungrouped data

Variance of ungrouped data

$$\text{Var}(\sigma^2) = \frac{1}{n} \sum (x_i - \bar{x})^2$$

Standard Deviation (S.D.) for ungrouped data

$$\text{S.D.}(\sigma) = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$$

4.

Coefficient of Variation

Coefficient of variation (C.V.)

$$= \frac{\sigma}{\bar{x}} \times 100; \bar{x} \neq 0$$

where $\bar{x}$ is mean

5.

Range

The difference between the highest and the lowest element of a data called its range.

$$\text{range} = x_{\max} - x_{\min}$$

Eg: Find the range of the given data:

4, 7, 8, 9, 10, 12, 13, 17

Here, $x_{\max} = 17$ and $x_{\min} = 4$

Range = $17 - 4 = 13$

6.

Mean Deviation

1. Mean Deviation (M.D.) for ungrouped data

M.D. about mean i.e., $\text{M.D.}(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}$, where $\bar{x}$ is mean and n = no. of terms

M.D. about median i.e., $\text{M.D.}(M) = \frac{\sum |x_i - M|}{n}$

n = no. of terms

2. Mean Deviation (M.D.) for grouped data

M.D. about mean i.e., $\text{M.D.}(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{N}$ where $N = \sum f_i$

M.D. about median i.e., $\text{M.D.}(M) = \frac{1}{N} \sum f_i |x_i - M|$

7.

Variance & Standard Deviation of a Continuous frequency distribution

Variance of a continuous frequency distribution

$\text{Var}(\sigma^2) = \frac{1}{N} \sum f_i (x_i - \bar{x})^2$ standard deviation of a continuous frequency distribution $\text{S.D}(\sigma) = \frac{1}{N} \sqrt{N \sum f_i x_i^2 - (\sum f_i x_i)^2}$

8.

Step Deviation or Shortcut Method

$$\text{M.D.}(\bar{x}) = \frac{1}{N} \sum f_i |x_i - \bar{x}|$$

$$\text{where } \bar{x} = a + \frac{\sum f_i d_i}{N} \times h$$

$$\text{and } d_i = \frac{x_i - a}{h}$$

Here, a = assumed mean

h = common factor

N = sum of frequencies

$$\text{M.D.}(M) = \frac{1}{N} \sum f_i |x_i - M|$$

$$\text{where, } M(\text{median}) = l + \frac{\frac{N}{2} - C}{f} \times h$$

9.

Shortcut method to find variance and standard deviation

$$\text{Variance } (\sigma^2) = \frac{h^2}{N^2} \left[N \sum f_i y_i^2 - (\sum f_i y_i)^2 \right]$$

$$\text{Standard Deviation } \sigma = \frac{h}{N} \sqrt{N \sum f_i y_i^2 - (\sum f_i y_i)^2}$$

$$\text{where } y_i = \frac{x_i - A}{h}$$

$$N = \sum f_i$$

A = assumed mean

h = width of class interval

MATHEMATICAL REASONING & PMI

Contrapositive and Converse Statements

- 01**
- The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$
 - The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$

Compound Statement

Many mathematical statements are obtained by combining one or more statements using some connecting words like "and", "or" etc. those statement are called a "Compound Statement."

Sentence

A sentence is called a mathematically acceptable statement if it is either true or false but not both.

Negation

A statement which is formed by changing the true value of a given statement by using the word like 'no', 'not' is called negation of given statement.

Implications

- 05**
- These are statements with word "if then", "only if" and "if and only if". if p then q is the same as following:
- p implies q is denoted by $p \Rightarrow q$, then symbol $\Rightarrow$ stands for implies
 - p is a sufficient condition for q , then symbol $\Rightarrow$
 - p only if q • q is a necessary condition for p
 - $\sim q$ implies $\sim p$

Truth Table for Logical Operations

Conjunction operation:			Disjunction operation:			Negation operation:		Implication operation:			Biconditional operation:		
p	q	$p \wedge q$	p	q	$p \vee q$	p	$\sim p$	p	q	$p \Rightarrow q$	p	q	$p \Leftrightarrow q$
T	T	T	T	T	T	T	F	T	T	T	T	T	T
T	F	F	T	F	T	T	T	T	F	F	T	F	F
F	T	F	F	T	T	F	T	F	T	T	F	T	F
F	F	F	F	F	F	F	F	F	F	T	F	F	T

Additional Important Points

- 07**
- $p \Rightarrow q = \sim p \vee q$
 - $\sim (p \Rightarrow q) = \sim (\sim p \vee q) = p \wedge (\sim q)$
 - $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$
 - $\sim (p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$
 - $(p \Leftrightarrow q) \Leftrightarrow r = p \Leftrightarrow (q \Leftrightarrow r)$

General Logical Equivalences

It comprises the following laws:

Idempotent Law

- $p \vee p \Leftrightarrow p$
- $p \wedge p \Leftrightarrow p$

Associative Law

- $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Commutative Law

- $p \vee q \Leftrightarrow q \vee p$
- $p \wedge q \Leftrightarrow q \wedge p$

Distributive Law

- $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
- $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Identity Laws

- $p \wedge T \Leftrightarrow p$
- $p \vee F \Leftrightarrow p$
- $p \wedge F \Leftrightarrow F$

Complement Laws

- $p \vee \sim p \Leftrightarrow T$
- $p \wedge \sim p \Leftrightarrow F$
- $\sim T \Leftrightarrow F$
- $\sim F \Leftrightarrow T$

Absorption Law

- $p \vee (p \wedge q) \Leftrightarrow p$
- $p \wedge (p \vee q) \Leftrightarrow p$

Involution Law

- $p \sim (\sim p) \Leftrightarrow p$

De-Morgan's Law

- $q \sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$
- $q \sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$

Tautology and Fallacy

- A tautology asserts that every possible interpretation has only one output, namely true.
- Fallacy implies an assertion of false in every possible interpretation.

NOTE: To evaluate tautology and fallacy, we can adapt the concept of the truth table that includes every possible valuation.

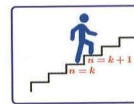
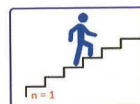
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \vee (q \Rightarrow p)$	$\sim \{(p \Rightarrow q) \vee (q \Rightarrow p)\}$
T	T	T	T	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

Principle of Mathematical Induction

Base Case: The given statement is correct for first natural number that is, for $n=1$, $p(1)$ is true.

Inductive Step: If the given statement is true for any natural number like $n=k$ then it will be correct for $n = k + 1$ also that is, if $p(k)$ is true then $p(k + 1)$ will also be true.

The first principle of mathematical induction says that if both the above steps are proven then $p(n)$ is true for all natural numbers.



PROPERTIES AND SOLUTION OF A TRIANGLE

01

Sine Rule

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in

$$\text{triangle } ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note :- The above rule can also be written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine Formulae

$$\text{In any } \triangle ABC, \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection Formulae

$$\text{In any } \triangle ABC, a = b \cos C + c \cos B, b = c \cos A + a \cos C, c = a \cos B + b \cos A$$

03 Area Of A Triangle

If Δ be the area of a triangle ABC, then

$$(i) \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$$

$$(ii) \Delta = \frac{1}{2} \frac{a^2 \sin B \sin C}{\sin(B+C)} = \frac{1}{2} \frac{b^2 \sin C \sin A}{\sin(C+A)} = \frac{1}{2} \frac{c^2 \sin A \sin B}{\sin(A+B)}$$

$$(iii) \Delta = \sqrt{s(s-a)(s-b)(s-c)} \text{ (Heron's formula)}$$

Form above results, we obtain following values of $\sin A$, $\sin B$, $\sin C$,

$$(iv) \sin A = \frac{2\Delta}{bc} = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$(v) \sin B = \frac{2\Delta}{ca} = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}$$

$$(vi) \sin C = \frac{2\Delta}{ab} = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}$$

07 Formulae For r_1, r_2, r_3

In any $\triangle ABC$, we have

$$(i) r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$$

$$(ii) r_1 = s \tan \frac{A}{2}, r_2 = s \tan \frac{B}{2}, r_3 = s \tan \frac{C}{2}$$

$$(iii) r_1 = a \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}, r_2 = b \frac{\cos \frac{C}{2} \cos \frac{A}{2}}{\cos \frac{B}{2}}, r_3 = c \frac{\cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}}$$

$$(iv) r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}, r_2 = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}, r_3 = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

02

Trigonometrical Ratios of Half of The Angles of A Triangle

In any $\triangle ABC$, we have

$$(i) \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}, \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$(ii) \cos \frac{A}{2} = \sqrt{\frac{s(s-b)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{s(s-c)}{ac}}, \cos \frac{C}{2} = \sqrt{\frac{s(s-a)}{ab}}$$

$$(iii) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

04

Napier's Analogy

In any triangle ABC,

$$\tan \left(\frac{B-C}{2} \right) = \frac{b-c}{b+c} \cot \frac{A}{2}$$

$$\tan \left(\frac{C-A}{2} \right) = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$\tan \left(\frac{A-B}{2} \right) = \frac{a-b}{a+b} \cot \frac{C}{2}$$

05

Circumcircle of A Triangle

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}$$

$$\text{Where } \Delta \text{ is area of triangle and } s = \frac{a+b+c}{2}$$

06

Incircle of A Triangle

$$(i) r = \frac{\Delta}{s}$$

$$(ii) r = (s-a) \tan \left(\frac{A}{2} \right), r = (s-b) \tan \left(\frac{B}{2} \right) \text{ and } r = (s-c) \tan \left(\frac{C}{2} \right)$$

$$(iii) r = \frac{a \sin \left(\frac{B}{2} \right) \sin \left(\frac{C}{2} \right)}{\cos \left(\frac{A}{2} \right)}, r = \frac{b \sin \left(\frac{A}{2} \right) \sin \left(\frac{C}{2} \right)}{\cos \left(\frac{B}{2} \right)} \text{ and } r = \frac{c \sin \left(\frac{B}{2} \right) \sin \left(\frac{A}{2} \right)}{\cos \left(\frac{C}{2} \right)}$$

$$(iv) r = 4R \sin \left(\frac{A}{2} \right) \cdot \sin \left(\frac{B}{2} \right) \cdot \sin \left(\frac{C}{2} \right)$$

08 Pedal Triangle

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre. Let the perpendicular AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of $\triangle ABC$, respectively, meet at O. Then O is the orthocentre of the $\triangle ABC$.

1. The triangle DEF is called the pedal triangle of the $\triangle ABC$.
2. The distances of the orthocentre from the vertices and the sides - If O is the orthocentre and DEF the pedal triangle of the $\triangle ABC$, where AD, BE, CF are the perpendiculars drawn from A, B, C on the opposite sides BC, CA, AB respectively, then

(i) $OA = 2R \cos A$, $OB = 2R \cos B$ and $OC = 2R \cos C$

(ii) $OD = 2R \cos B \cos C$, $OE = 2R \cos C \cos A$ and $OF = 2R \cos A \cos B$

(iii) The circumradius of the pedal triangle = $\frac{R}{2}$

(iv) The area of pedal triangle = $2\Delta \cos A \cos B \cos C$.

09 Some Important Results

(1) $\tan \frac{A}{2} \tan \frac{B}{2} = \frac{s-c}{s} \therefore \cot \frac{A}{2} \cot \frac{B}{2} = \frac{s}{s-c}$

(2) $\tan \frac{A}{2} + \tan \frac{B}{2} = \frac{c}{s} \cot \frac{C}{2} = \frac{c}{\Delta} (s-c)$

(3) $\tan \frac{A}{2} - \tan \frac{B}{2} = \frac{a-b}{\Delta} (s-c)$

(4) $\cot \frac{A}{2} + \cot \frac{B}{2} = \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} \tan \frac{B}{2}} = \frac{c}{s-c} \cot \frac{C}{2}$

10 Height and Distance

Terms Related to Height and Distance

01 Line of Sight: It is the straight line that is drawn from the eye of an observer to the point of an object which is to be viewed.

02 Horizontal Level: It is the horizontal line drawn from the eye of the viewer.

03 The angle of elevation: It is the angle formed between the line of sight and horizontal level if the object is above the horizontal level.

04 The Angle of Depression: It is the angle formed between the line of sight and the horizontal level if the object is below the horizontal level.

To calculate the angle of elevation or depression we can use the following formul

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}, \cos \theta = \frac{\text{Base}}{\text{Hypotenuse}}, \tan \theta = \frac{\text{Perpendicular}}{\text{Base}}$$