

CONTINUITY AND DIFFERENTIABILITY

5.1 Overview

5.1.1 Continuity of a function at a point

Let f be a real function on a subset of the real numbers and let c be a point in the domain of f . Then f is continuous at c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and are equal to each other, i.e.,

$$\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$$

then f is said to be continuous at $x = c$.

5.1.2 Continuity in an interval

- (i) f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
- (ii) f is said to be continuous in the closed interval $[a, b]$ if

- f is continuous in (a, b)
- $\lim_{x \rightarrow a^+} f(x) = f(a)$
- $\lim_{x \rightarrow b^-} f(x) = f(b)$

5.1.3 Geometrical meaning of continuity

- (i) Function f will be continuous at $x = c$ if there is no break in the graph of the function at the point $(c, f(c))$.
- (ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

5.1.4 Discontinuity

The function f will be discontinuous at $x = a$ in any of the following cases :

- (i) $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist but are not equal.
- (ii) $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist and are equal but not equal to $f(a)$.
- (iii) $f(a)$ is not defined.

5.1.5 Continuity of some of the common functions

Function $f(x)$	Interval in which f is continuous
1. The constant function, i.e. $f(x) = c$	R
2. The identity function, i.e. $f(x) = x$	
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$	
4. $ x - a $	$(-\infty, \infty)$
5. x^n , n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	R - $\{x : q(x) = 0\}$
7. $\sin x, \cos x$	R
8. $\tan x, \sec x$	R - $\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x, \operatorname{cosec} x$	R - $\{n\pi : n \in \mathbf{Z}\}$

- | | |
|--|-----------------------------|
| 10. e^x | R |
| 11. $\log x$ | $(0, \infty)$ |
| 12. The inverse trigonometric functions,
i.e., $\sin^{-1} x$, $\cos^{-1} x$ etc. | In their respective domains |

5.1.6 Continuity of composite functions

Let f and g be real valued functions such that (fog) is defined at a . If g is continuous at a and f is continuous at $g(a)$, then (fog) is continuous at a .

5.1.7 Differentiability

The function defined by $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x . In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$, called left hand derivative, denoted by $Lf'(c)$, and $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$, called right hand derivative, denoted by $Rf'(c)$, are finite and equal.

- (i) The function $y = f(x)$ is said to be differentiable in an open interval (a, b) if it is differentiable at every point of (a, b)
- (ii) The function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$ if $Rf'(a)$ and $Lf'(b)$ exist and $f'(x)$ exists for every point of (a, b) .
- (iii) Every differentiable function is continuous, but the converse is not true

5.1.8 Algebra of derivatives

If u, v are functions of x , then

$$(i) \quad \frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$(ii) \quad \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$(iii) \quad \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

5.1.9 Chain rule is a rule to differentiate composition of functions. Let $f = v \circ u$. If

$$t = u(x) \text{ and both } \frac{dt}{dx} \text{ and } \frac{dv}{dt} \text{ exist then } \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

5.1.10 Following are some of the standard derivatives (in appropriate domains)

$$1. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad 2. \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad 4. \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$$

$$6. \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$$

5.1.11 Exponential and logarithmic functions

- (i) The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$. Its domain is $\mathbf{R}$, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.
- (ii) Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$. Logarithm of a to the base b is denoted by $\log_b a$. If the base $b = 10$, we say it is common logarithm and if $b = e$, then we say it is natural logarithms. $\log x$ denotes the logarithm function to base e . The domain of logarithm function is $\mathbf{R}^+$, the set of all positive real numbers and the range is the set of all real numbers.
- (iii) The properties of logarithmic function to any base $b > 1$ are listed below:

$$1. \log_b(xy) = \log_b x + \log_b y$$

$$2. \log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$$

$$3. \log_b x^n = n \log_b x$$

$$4. \log_b x = \frac{\log_c x}{\log_c b}, \text{ where } c > 1$$

$$5. \log_b x = \frac{1}{\log_x b}$$

$$6. \log_b b = 1 \text{ and } \log_b 1 = 0$$

(iv) The derivative of e^x w.r.t., x is e^x , i.e. $\frac{d}{dx}(e^x) = e^x$. The derivative of $\log x$

w.r.t., x is $\frac{1}{x}$; i.e. $\frac{d}{dx}(\log x) = \frac{1}{x}$.

5.1.12 Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = (u(x))^{v(x)}$, where both f and u need to be positive functions for this technique to make sense.

5.1.13 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x , then to find derivative of $f(x)$ w.r.t.

to $g(x)$, i.e., to find $\frac{du}{dv}$, we use the formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}.$$

5.1.14 Second order derivative

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is called the second order derivative of y w.r.t. x . It is denoted by y'' or y_2 , if $y = f(x)$.

5.1.15 Rolle's Theorem

Let $f: [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Geometrically Rolle's theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to x-axis (abscissa of the point lying in (a, b)).

5.1.16 Mean Value Theorem (Lagrange)

Let $f: [a, b] \rightarrow \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b) . Then

there exists at least one point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Geometrically, Mean Value Theorem states that there exists at least one point c in (a, b) such that the tangent at the point $(c, f(c))$ is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.

5.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the value of the constant k so that the function f defined below is

continuous at $x = 0$, where $f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$.

Solution It is given that the function f is continuous at $x = 0$. Therefore, $\lim_{x \rightarrow 0} f(x) = f(0)$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{8x^2} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin^2 2x}{8x^2} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right)^2 = k$$

$$\Rightarrow k = 1$$

Thus, f is continuous at $x = 0$ if $k = 1$.

Example 2 Discuss the continuity of the function $f(x) = \sin x \cdot \cos x$.

Solution Since $\sin x$ and $\cos x$ are continuous functions and product of two continuous function is a continuous function, therefore $f(x) = \sin x \cdot \cos x$ is a continuous function.

Example 3 If $f(x) = \begin{cases} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$ is continuous at $x = 2$, find

the value of k .

Solution Given $f(2) = k$.

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} \\ &= \lim_{x \rightarrow 2} \frac{(x+5)(x-2)^2}{(x-2)^2} = \lim_{x \rightarrow 2} (x+5) = 7 \end{aligned}$$

As f is continuous at $x = 2$, we have

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= f(2) \\ \Rightarrow k &= 7. \end{aligned}$$

Example 4 Show that the function f defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution Left hand limit at $x = 0$ is given by

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0 \quad \left[\text{since, } -1 < \sin \frac{1}{x} < 1 \right]$$

Similarly $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = 0$. Moreover $f(0) = 0$.

Thus $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$. Hence f is continuous at $x = 0$

Example 5 Given $f(x) = \frac{1}{x-1}$. Find the points of discontinuity of the composite function $y = f[f(x)]$.

Solution We know that $f(x) = \frac{1}{x-1}$ is discontinuous at $x = 1$

Now, for $x \neq 1$,

$$f(f(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}-1} = \frac{x-1}{2-x},$$

which is discontinuous at $x = 2$.

Hence, the points of discontinuity are $x = 1$ and $x = 2$.

Example 6 Let $f(x) = x|x|$, for all $x \in \mathbf{R}$. Discuss the derivability of $f(x)$ at $x = 0$

Solution We may rewrite f as $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$\text{Now } Lf'(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-h^2 - 0}{h} = \lim_{h \rightarrow 0^-} -h = 0$$

$$\text{Now } Rf'(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^2 - 0}{h} = \lim_{h \rightarrow 0^+} h = 0$$

Since the left hand derivative and right hand derivative both are equal, hence f is differentiable at $x = 0$.

Example 7 Differentiate $\sqrt{\tan \sqrt{x}}$ w.r.t. x

Solution Let $y = \sqrt{\tan \sqrt{x}}$. Using chain rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \cdot \frac{d}{dx}(\tan \sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \cdot \sec^2 \sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} (\sec^2 \sqrt{x}) \left(\frac{1}{2\sqrt{x}} \right) \\ &= \frac{(\sec^2 \sqrt{x})}{4\sqrt{x}\sqrt{\tan \sqrt{x}}}. \end{aligned}$$

Example 8 If $y = \tan(x+y)$, find $\frac{dy}{dx}$.

Solution Given $y = \tan(x+y)$. differentiating both sides w.r.t. x , we have

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(x+y) \frac{d}{dx}(x+y) \\ &= \sec^2(x+y) \left(1 + \frac{dy}{dx}\right)\end{aligned}$$

$$\text{or} \quad [1 - \sec^2(x+y)] \frac{dy}{dx} = \sec^2(x+y)$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{\sec^2(x+y)}{1 - \sec^2(x+y)} = -\operatorname{cosec}^2(x+y).$$

Example 9 If $e^x + e^y = e^{x+y}$, prove that

$$\frac{dy}{dx} = -e^{y-x}.$$

Solution Given that $e^x + e^y = e^{x+y}$. Differentiating both sides w.r.t. x , we have

$$e^x + e^y \frac{dy}{dx} = e^{x+y} \left(1 + \frac{dy}{dx}\right)$$

$$\text{or} \quad (e^y - e^{x+y}) \frac{dy}{dx} = e^{x+y} - e^x,$$

$$\text{which implies that } \frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}} = \frac{e^x + e^y - e^x}{e^y - e^x - e^y} = -e^{y-x}.$$

Example 10 Find $\frac{dy}{dx}$, if $y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Solution Put $x = \tan \theta$, where $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$.

$$\begin{aligned}\text{Therefore, } y &= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \\ &= \tan^{-1}(\tan 3\theta) \\ &= 3\theta \quad \left(\text{because } -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \right) \\ &= 3 \tan^{-1} x\end{aligned}$$

Hence, $\frac{dy}{dx} = \frac{3}{1+x^2}$.

Example 11 If $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

Solution We have $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$, where $0 < x < 1$.

Put $x = \sin A$ and $\sqrt{x} = \sin B$

Therefore, $y = \sin^{-1}\{\sin A\sqrt{1-\sin^2 B} - \sin B\sqrt{1-\sin^2 A}\}$

$$= \sin^{-1}\{\sin A \cos B - \sin B \cos A\}$$

$$= \sin^{-1}\{\sin(A-B)\} = A - B$$

Thus $y = \sin^{-1} x - \sin^{-1} \sqrt{x}$

Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x}) \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}. \end{aligned}$$

Example 12 If $x = a \sec^3 \theta$ and $y = a \tan^3 \theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Solution We have $x = a \sec^3 \theta$ and $y = a \tan^3 \theta$.

Differentiating w.r.t. θ , we get

$$\frac{dx}{d\theta} = 3a \sec^2 \theta \frac{d}{d\theta}(\sec \theta) = 3a \sec^3 \theta \tan \theta$$

and $\frac{dy}{d\theta} = 3a \tan^2 \theta \frac{d}{d\theta}(\tan \theta) = 3a \tan^2 \theta \sec^2 \theta.$

Thus $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \tan^2 \theta \sec^2 \theta}{3a \sec^3 \theta \tan \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta.$

Hence, $\left(\frac{dy}{dx}\right)_{at \theta = \frac{\pi}{3}} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Example 13 If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution We have $x^y = e^{x-y}$. Taking logarithm on both sides, we get

$$\begin{aligned} y \log x &= x - y \\ \Rightarrow y(1 + \log x) &= x \end{aligned}$$

$$\text{i.e. } y = \frac{x}{1 + \log x}$$

Differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot 1 - x \left(\frac{1}{x}\right)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}.$$

Example 14 If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1 - \sin x)^2}$.

Solution We have $y = \tan x + \sec x$. Differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 x + \sec x \tan x \\ &= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x} = \frac{1 + \sin x}{(1 + \sin x)(1 - \sin x)}. \end{aligned}$$

$$\text{thus } \frac{dy}{dx} = \frac{1}{1 - \sin x}.$$

Now, differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{-(-\cos x)}{(1 - \sin x)^2} = \frac{\cos x}{(1 - \sin x)^2}$$

Example 15 If $f(x) = |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Solution When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$ so that $|\cos x| = -\cos x$, i.e., $f(x) = -\cos x$
 $\Rightarrow f'(x) = \sin x$.

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Example 16 If $f(x) = |\cos x - \sin x|$, find $f'\left(\frac{\pi}{6}\right)$.

Solution When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so that $\cos x - \sin x > 0$, i.e.,

$$f(x) = \cos x - \sin x$$

$$\Rightarrow f'(x) = -\sin x - \cos x$$

Hence $f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1+\sqrt{3})$.

Example 17 Verify Rolle's theorem for the function, $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$. Note that:

- (i) The function f is continuous in $\left[0, \frac{\pi}{2}\right]$, as f is a sine function, which is always continuous.
- (ii) $f'(x) = 2\cos 2x$, exists in $\left(0, \frac{\pi}{2}\right)$, hence f is derivable in $\left(0, \frac{\pi}{2}\right)$.
- (iii) $f(0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}\right) = \sin \pi = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Conditions of Rolle's theorem are satisfied. Hence there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Thus

$$2 \cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}.$$

Example 18 Verify mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ in $[3, 5]$.

Solution (i) Function f is continuous in $[3, 5]$ as product of polynomial functions is a polynomial, which is continuous.

(ii) $f'(x) = 3x^2 - 36x + 99$ exists in $(3, 5)$ and hence derivable in $(3, 5)$.

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one $c \in (3, 5)$ such that

$$\begin{aligned} f'(c) &= \frac{f(5) - f(3)}{5 - 3} \\ \Rightarrow 3c^2 - 36c + 99 &= \frac{8 - 0}{2} = 4 \\ \Rightarrow c &= 6 \pm \sqrt{\frac{13}{3}}. \end{aligned}$$

Hence $c = 6 - \sqrt{\frac{13}{3}}$ (since other value is not permissible).

Long Answer (L.A.)

Example 19 If $f(x) = \frac{\sqrt{2} \cos x - 1}{\cot x - 1}, x \neq \frac{\pi}{4}$

find the value of $f\left(\frac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \frac{\pi}{4}$.

Solution Given, $f(x) = \frac{\sqrt{2} \cos x - 1}{\cot x - 1}, x \neq \frac{\pi}{4}$

$$\begin{aligned} \text{Therefore, } \lim_{x \rightarrow \frac{\pi}{4}} f(x) &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} \cos x - 1}{\cot x - 1} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sqrt{2} \cos x - 1) \sin x}{\cos x - \sin x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sqrt{2} \cos x - 1)}{(\sqrt{2} \cos x + 1)} \cdot \frac{(\sqrt{2} \cos x + 1)}{(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2} \cos x + 1} \cdot (\sin x) \\
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left(\frac{\cos x + \sin x}{\sqrt{2} \cos x + 1} \right) (\sin x) \\
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\cos x + \sin x)}{\sqrt{2} \cos x + 1} \sin x \\
&= \frac{\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}
\end{aligned}$$

Thus, $\lim_{x \rightarrow \frac{\pi}{4}} f(x) = \frac{1}{2}$

If we define $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, then $f(x)$ will become continuous at $x = \frac{\pi}{4}$. Hence for f to be continuous at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

Example 20 Show that the function f given by $f(x) = \begin{cases} \frac{1}{e^x - 1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

is discontinuous at $x = 0$.

Solution The left hand limit of f at $x = 0$ is given by

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1} = \frac{0 - 1}{0 + 1} = -1$$

Similarly,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x \rightarrow 0^+} \frac{1 - \frac{1}{e^x}}{1 + \frac{1}{e^x}} = \lim_{x \rightarrow 0^+} \frac{1 - e^{-x}}{1 + e^{-x}} = \frac{1 - 0}{1 + 0} = 1$$

Thus $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$, therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist. Hence f is discontinuous at $x = 0$.

Example 21 Let $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{if } x > 0 \end{cases}$

For what value of a , f is continuous at $x = 0$?

Solution Here $f(0) = a$ Left hand limit of f at 0 is

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{1 - \cos 4x}{x^2} = \lim_{x \rightarrow 0^-} \frac{2 \sin^2 2x}{x^2} \\ &= \lim_{2x \rightarrow 0^-} 8 \left(\frac{\sin 2x}{2x} \right)^2 = 8 (1)^2 = 8. \end{aligned}$$

and right hand limit of f at 0 is

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{(\sqrt{16 + \sqrt{x}} + 4)(\sqrt{16 + \sqrt{x}} - 4)} \end{aligned}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = \lim_{x \rightarrow 0^+} (\sqrt{16+\sqrt{x}}+4) = 8$$

Thus, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 8$. Hence f is continuous at $x = 0$ only if $a = 8$.

Example 22 Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } -3 \leq x < -2 \\ x+1, & \text{if } -2 \leq x < 0 \\ x+2, & \text{if } 0 \leq x \leq 1 \end{cases}$$

Solution The only doubtful points for differentiability of $f(x)$ are $x = -2$ and $x = 0$.
Differentiability at $x = -2$.

$$\begin{aligned} \text{Now } L f'(-2) &= \lim_{h \rightarrow 0^-} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{2(-2+h)+3 - (-2+1)}{h} = \lim_{h \rightarrow 0^-} \frac{2h}{h} = \lim_{h \rightarrow 0^-} 2 = 2. \end{aligned}$$

$$\begin{aligned} \text{and } R f'(-2) &= \lim_{h \rightarrow 0^+} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{-2+h+1 - (-2+1)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h-1 - (-1)}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

Thus $R f'(-2) \neq L f'(-2)$. Therefore f is not differentiable at $x = -2$.
Similarly, for differentiability at $x = 0$, we have

$$\begin{aligned} L(f'(0)) &= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{0+h+1 - (0+2)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{h-1}{h} = \lim_{h \rightarrow 0^-} \left(1 - \frac{1}{h}\right) \end{aligned}$$

which does not exist. Hence f is not differentiable at $x = 0$.

Example 23 Differentiate $\tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$ with respect to $\cos^{-1} (2x\sqrt{1-x^2})$, where $x \in \left(\frac{1}{\sqrt{2}}, 1 \right)$.

Solution Let $u = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$ and $v = \cos^{-1} (2x\sqrt{1-x^2})$.

We want to find $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Now $u = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right)$. Put $x = \sin \theta$. $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2} \right)$.

Then $u = \tan^{-1} \left(\frac{\sqrt{1-\sin^2 \theta}}{\sin \theta} \right) = \tan^{-1} (\cot \theta)$
 $= \tan^{-1} \left\{ \tan \left(\frac{\pi}{2} - \theta \right) \right\} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1} x$

Hence $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

Now $v = \cos^{-1} (2x\sqrt{1-x^2})$
 $= \frac{\pi}{2} - \sin^{-1} (2x\sqrt{1-x^2})$
 $= \frac{\pi}{2} - \sin^{-1} (2\sin \theta \sqrt{1-\sin^2 \theta}) = \frac{\pi}{2} - \sin^{-1} (\sin 2\theta)$
 $= \frac{\pi}{2} - \sin^{-1} \{ \sin (\pi - 2\theta) \}$ [since $\frac{\pi}{2} < 2\theta < \pi$]

$$= \frac{\pi}{2} - (\pi - 2\theta) = \frac{-\pi}{2} + 2\theta$$

$$\Rightarrow v = \frac{-\pi}{2} + 2\sin^{-1}x$$

$$\Rightarrow \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

$$\text{Hence} \quad \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{-1}{2}.$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 24 to 35.

Example 24 The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$, then the value of k is

- | | |
|-------|---------|
| (A) 3 | (B) 2 |
| (C) 1 | (D) 1.5 |

Solution (B) is the Correct answer.

Example 25 The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at

- | | |
|-------|---------|
| (A) 4 | (B) -2 |
| (C) 1 | (D) 1.5 |

Solution (D) is the correct answer. The greatest integer function $[x]$ is discontinuous at all integral values of x . Thus D is the correct answer.

Example 26 The number of points at which the function $f(x) = \frac{1}{x - [x]}$ is not continuous is

- | | |
|-------|-------------------|
| (A) 1 | (B) 2 |
| (C) 3 | (D) none of these |

Solution (D) is the correct answer. As $x - [x] = 0$, when x is an integer so $f(x)$ is discontinuous for all $x \in \mathbf{Z}$.

Example 27 The function given by $f(x) = \tan x$ is discontinuous on the set

- (A) $\{n\pi : n \in \mathbf{Z}\}$ (B) $\{2n\pi : n \in \mathbf{Z}\}$
 (C) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$ (D) $\left\{\frac{n\pi}{2} : n \in \mathbf{Z}\right\}$

Solution C is the correct answer.

Example 28 Let $f(x) = |\cos x|$. Then,

- (A) f is everywhere differentiable.
 (B) f is everywhere continuous but not differentiable at $x = n\pi, n \in \mathbf{Z}$.
 (C) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}$.
 (D) none of these.

Solution C is the correct answer.

Example 29 The function $f(x) = |x| + |x-1|$ is

- (A) continuous at $x = 0$ as well as at $x = 1$.
 (B) continuous at $x = 1$ but not at $x = 0$.
 (C) discontinuous at $x = 0$ as well as at $x = 1$.
 (D) continuous at $x = 0$ but not at $x = 1$.

Solution Correct answer is A.

Example 30 The value of k which makes the function defined by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}, \text{ continuous at } x = 0 \text{ is}$$

(A) 8 (B) 1
 (C) -1 (D) none of these

Solution (D) is the correct answer. Indeed $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

Example 31 The set of points where the functions f given by $f(x) = |x-3| \cos x$ is differentiable is

- (A) $\mathbf{R}$ (B) $\mathbf{R} - \{3\}$
 (C) $(0, \infty)$ (D) none of these

Solution B is the correct answer.

Example 32 Differential coefficient of $\sec(\tan^{-1}x)$ w.r.t. x is

- (A) $\frac{x}{\sqrt{1+x^2}}$ (B) $\frac{x}{1+x^2}$
 (C) $x\sqrt{1+x^2}$ (D) $\frac{1}{\sqrt{1+x^2}}$

Solution (A) is the correct answer.

Example 33 If $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $\frac{du}{dv}$ is

- (A) $\frac{1}{2}$ (B) x (C) $\frac{1-x^2}{1+x^2}$ (D) 1

Solution (D) is the correct answer.

Example 34 The value of c in Rolle's Theorem for the function $f(x) = e^x \sin x$, $x \in [0, \pi]$ is

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

Solution (D) is the correct answer.

Example 35 The value of c in Mean value theorem for the function $f(x) = x(x-2)$, $x \in [1, 2]$ is

- (A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$

Solution (A) is the correct answer.

Example 36 Match the following

COLUMN-I

COLUMN-II

(A) If a function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{if } x \neq 0 \\ \frac{k}{2}, & \text{if } x = 0 \end{cases}$

(a) $|x|$

is continuous at $x = 0$, then k is equal to

- (B) Every continuous function is differentiable (b) True
 (C) An example of a function which is continuous everywhere but not differentiable at exactly one point (c) 6
 (D) The identity function i.e. $f(x) = x \forall x \in \mathbb{R}$ is a continuous function (d) False

Solution $A \rightarrow c, B \rightarrow d, C \rightarrow a, D \rightarrow b$

Fill in the blanks in each of the Examples 37 to 41.

Example 37 The number of points at which the function $f(x) = \frac{1}{\log|x|}$ is discontinuous is _____.

Solution The given function is discontinuous at $x = 0, \pm 1$ and hence the number of points of discontinuity is 3.

Example 38 If $f(x) = \begin{cases} ax+1 & \text{if } x \geq 1 \\ x+2 & \text{if } x < 1 \end{cases}$ is continuous, then a should be equal to _____.

Solution $a = 2$

Example 39 The derivative of $\log_{10} x$ w.r.t. x is _____.

Solution $(\log_{10} e) \frac{1}{x}$.

Example 40 If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to _____.

Solution 0.

Example 41 The derivative of $\sin x$ w.r.t. $\cos x$ is _____.

Solution $-\cot x$

State whether the statements are True or False in each of the Exercises 42 to 46.

Example 42 For continuity, at $x = a$, each of $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ is equal to $f(a)$.

Solution True.

Example 43 $y = |x - 1|$ is a continuous function.

Solution True.

Example 44 A continuous function can have some points where limit does not exist.

Solution False.

Example 45 $|\sin x|$ is a differentiable function for every value of x .

Solution False.

Example 46 $\cos |x|$ is differentiable everywhere.

Solution True.

5.3 EXERCISE

Short Answer (S.A.)

1. Examine the continuity of the function

$$f(x) = x^3 + 2x^2 - 1 \text{ at } x = 1$$

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x) = \begin{cases} 3x+5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$

at $x=2$

3. $f(x) = \begin{cases} \frac{1 - \cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$

at $x=0$

4. $f(x) = \begin{cases} \frac{2x^2 - 3x - 2}{x - 2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$

at $x=2$

5. $f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$

at $x=4$

6. $f(x) = \begin{cases} |x| \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

at $x=0$

7. $f(x) = \begin{cases} |x-a| \sin \frac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$

at $x=a$

8. $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

at $x=0$

9. $f(x) = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \frac{3}{2}, & \text{if } 1 < x \leq 2 \end{cases}$

at $x=1$

10. $f(x) = |x| + |x-1|$ at $x=1$

Find the value of k in each of the Exercises 11 to 14 so that the function f is continuous at the indicated point:

$$11. f(x) = \begin{cases} 3x-8, & \text{if } x \leq 5 \\ 2k, & \text{if } x > 5 \end{cases} \quad \text{at } x=5 \qquad 12. f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \quad \text{at } x=2$$

$$13. f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x \leq 1 \end{cases} \quad \text{at } x=0$$

$$14. f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \frac{1}{2}, & \text{if } x = 0 \end{cases} \quad \text{at } x=0$$

15. Prove that the function f defined by

$$f(x) = \begin{cases} \frac{x}{|x|+2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k .

16. Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a+b, & \text{if } x = 4 \\ \frac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at $x = 4$.

17. Given the function $f(x) = \frac{1}{x+2}$. Find the points of discontinuity of the composite function $y = f(f(x))$.

18. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

19. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

Examine the differentiability of f , where f is defined by

20.
$$f(x) = \begin{cases} x[x], & \text{if } 0 \leq x < 2 \\ (x-1)x, & \text{if } 2 \leq x < 3 \end{cases}$$

at $x = 2$.

21.
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{if } x \neq 0 \\ 0 & , \text{if } x = 0 \end{cases}$$

at $x = 0$.

22.
$$f(x) = \begin{cases} 1+x & , \text{if } x \leq 2 \\ 5-x & , \text{if } x > 2 \end{cases}$$

at $x = 2$.

23. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.

24. A function $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43) :

25. $2^{\cos^2 x}$

26. $\frac{8^x}{x^8}$

27. $\log(x + \sqrt{x^2 + a})$

28. $\log[\log(\log x^5)]$

29. $\sin \sqrt{x} + \cos^2 \sqrt{x}$

30. $\sin^n(ax^2 + bx + c)$

31. $\cos(\tan \sqrt{x+1})$

32. $\sin x^2 + \sin^2 x + \sin^2(x^2)$

33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

34. $(\sin x)^{\cos x}$

35. $\sin^m x \cdot \cos^n x$

36. $(x+1)^2(x+2)^3(x+3)^4$

$$37. \cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{-\pi}{4} < x < \frac{\pi}{4} \qquad 38. \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$$

$$39. \tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$40. \tan^{-1}\left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } \frac{a}{b} \tan x > -1$$

$$41. \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right), 0 < x < \frac{1}{\sqrt{2}}$$

$$42. \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$$

$$43. \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0$$

Find $\frac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48.

$$44. x = t + \frac{1}{t}, y = t - \frac{1}{t} \qquad 45. x = e^{\theta} \left(\theta + \frac{1}{\theta} \right), y = e^{-\theta} \left(\theta - \frac{1}{\theta} \right)$$

$$46. x = 3\cos\theta - 2\cos^3\theta, y = 3\sin\theta - 2\sin^3\theta.$$

$$47. \sin x = \frac{2t}{1+t^2}, \tan y = \frac{2t}{1-t^2}.$$

$$48. x = \frac{1+\log t}{t^2}, y = \frac{3+2\log t}{t}.$$

$$49. \text{ If } x = e^{\cos 2t} \text{ and } y = e^{\sin 2t}, \text{ prove that } \frac{dy}{dx} = \frac{-y \log x}{x \log y}.$$

$$50. \text{ If } x = a \sin 2t (1 + \cos 2t) \text{ and } y = b \cos 2t (1 - \cos 2t), \text{ show that } \left(\frac{dy}{dx} \right)_{\text{at } t = \frac{\pi}{4}} = \frac{b}{a}.$$

$$51. \text{ If } x = 3 \sin t - \sin 3t, y = 3 \cos t - \cos 3t, \text{ find } \frac{dy}{dx} \text{ at } t = \frac{\pi}{3}.$$

52. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.

53. Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ w.r.t. $\tan^{-1} x$ when $x \neq 0$.

Find $\frac{dy}{dx}$ when x and y are connected by the relation given in each of the Exercises 54 to 57.

54. $\sin(xy) + \frac{x}{y} = x^2 - y$

55. $\sec(x+y) = xy$

56. $\tan^{-1}(x^2 + y^2) = a$

57. $(x^2 + y^2)^2 = xy$

58. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$.

59. If $x = e^{\frac{x}{y}}$, prove that $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

60. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

61. If $y = (\cos x)^{(\cos x)^{(\cos x) \dots \infty}}$, show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$.

62. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

63. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

64. If $y = \tan^{-1} x$, find $\frac{d^2 y}{dx^2}$ in terms of y alone.

Verify the Rolle's theorem for each of the functions in Exercises 65 to 69.

65. $f(x) = x(x-1)^2$ in $[0, 1]$.

66. $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

67. $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$.

68. $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$.

69. $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$.

70. Discuss the applicability of Rolle's theorem on the function given by

$$f(x) = \begin{cases} x^2 + 1, & \text{if } 0 \leq x \leq 1 \\ 3 - x, & \text{if } 1 \leq x \leq 2 \end{cases}$$

71. Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to x -axis.

72. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to x -axis.

Verify mean value theorem for each of the functions given Exercises 73 to 76.

73. $f(x) = \frac{1}{4x-1}$ in $[1, 4]$.

74. $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$.

75. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

76. $f(x) = \sqrt{25-x^2}$ in $[1, 5]$.

77. Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the chord joining the points $(3, 0)$ and $(4, 1)$.

78. Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB . Also, find that point.

Long Answer (L.A.)

79. Find the values of p and q so that

$$f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \leq 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

80. If $x^m \cdot y^n = (x + y)^{m+n}$, prove that

(i) $\frac{dy}{dx} = \frac{y}{x}$ and (ii) $\frac{d^2y}{dx^2} = 0$.

81. If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

82. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

83. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function
- (A) $f(x) + g(x)$ (B) $f(x) - g(x)$
- (C) $f(x) \cdot g(x)$ (D) $\frac{g(x)}{f(x)}$

84. The function $f(x) = \frac{4-x^2}{4x-x^3}$ is
- (A) discontinuous at only one point
- (B) discontinuous at exactly two points
- (C) discontinuous at exactly three points
- (D) none of these

85. The set of points where the function f given by $f(x) = |2x-1| \sin x$ is differentiable is
- (A) $\mathbf{R}$ (B) $\mathbf{R} - \left\{\frac{1}{2}\right\}$

- (C) $(0, \infty)$ (D) none of these
86. The function $f(x) = \cot x$ is discontinuous on the set
 (A) $\{x = n\pi : n \in \mathbf{Z}\}$ (B) $\{x = 2n\pi : n \in \mathbf{Z}\}$
 (C) $\left\{x = (2n+1)\frac{\pi}{2} ; n \in \mathbf{Z}\right\}$ (iv) $\left\{x = \frac{n\pi}{2} ; n \in \mathbf{Z}\right\}$
87. The function $f(x) = e^{|x|}$ is
 (A) continuous everywhere but not differentiable at $x = 0$
 (B) continuous and differentiable everywhere
 (C) not continuous at $x = 0$
 (D) none of these.
88. If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is
 (A) 0 (B) -1
 (C) 1 (D) none of these
89. If $f(x) = \begin{cases} mx+1, & \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$, then
 (A) $m = 1, n = 0$ (B) $m = \frac{n\pi}{2} + 1$
 (C) $n = \frac{m\pi}{2}$ (D) $m = n = \frac{\pi}{2}$
90. Let $f(x) = |\sin x|$. Then
 (A) f is everywhere differentiable
 (B) f is everywhere continuous but not differentiable at $x = n\pi, n \in \mathbf{Z}$.
 (C) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}$.
 (D) none of these
91. If $y = \log \left(\frac{1-x^2}{1+x^2} \right)$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{4x^3}{1-x^4}$

(B) $\frac{-4x}{1-x^4}$

(C) $\frac{1}{4-x^4}$

(D) $\frac{-4x^3}{1-x^4}$

92. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{\cos x}{2y-1}$

(B) $\frac{\cos x}{1-2y}$

(C) $\frac{\sin x}{1-2y}$

(D) $\frac{\sin x}{2y-1}$

93. The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1}x$ is

(A) 2

(B) $\frac{-1}{2\sqrt{1-x^2}}$

(C) $\frac{2}{x}$

(D) $1 - x^2$

94. If $x = t^2, y = t^3$, then $\frac{d^2y}{dx^2}$ is

(A) $\frac{3}{2}$

(B) $\frac{3}{4t}$

(C) $\frac{3}{2t}$

(D) $\frac{3}{4}$

95. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(A) 1

(B) -1

(C) $\frac{3}{2}$

(D) $\frac{1}{3}$

96. For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for mean value theorem is

(A) 1

(B) $\sqrt{3}$

(C) 2

(D) none of these

Fill in the blanks in each of the Exercises 97 to 101:

97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is _____.

98. Derivative of x^2 w.r.t. x^3 is _____.

99. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right) = \underline{\hspace{2cm}}$.

100. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right) = \underline{\hspace{2cm}}$.

101. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is _____.

State **True** or **False** for the statements in each of the Exercises 102 to 106.

102. Rolle's theorem is applicable for the function $f(x) = |x - 1|$ in $[0, 2]$.

103. If f is continuous on its domain D , then $|f|$ is also continuous on D .

104. The composition of two continuous function is a continuous function.

105. Trigonometric and inverse - trigonometric functions are differentiable in their respective domain.

106. If $f \cdot g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

